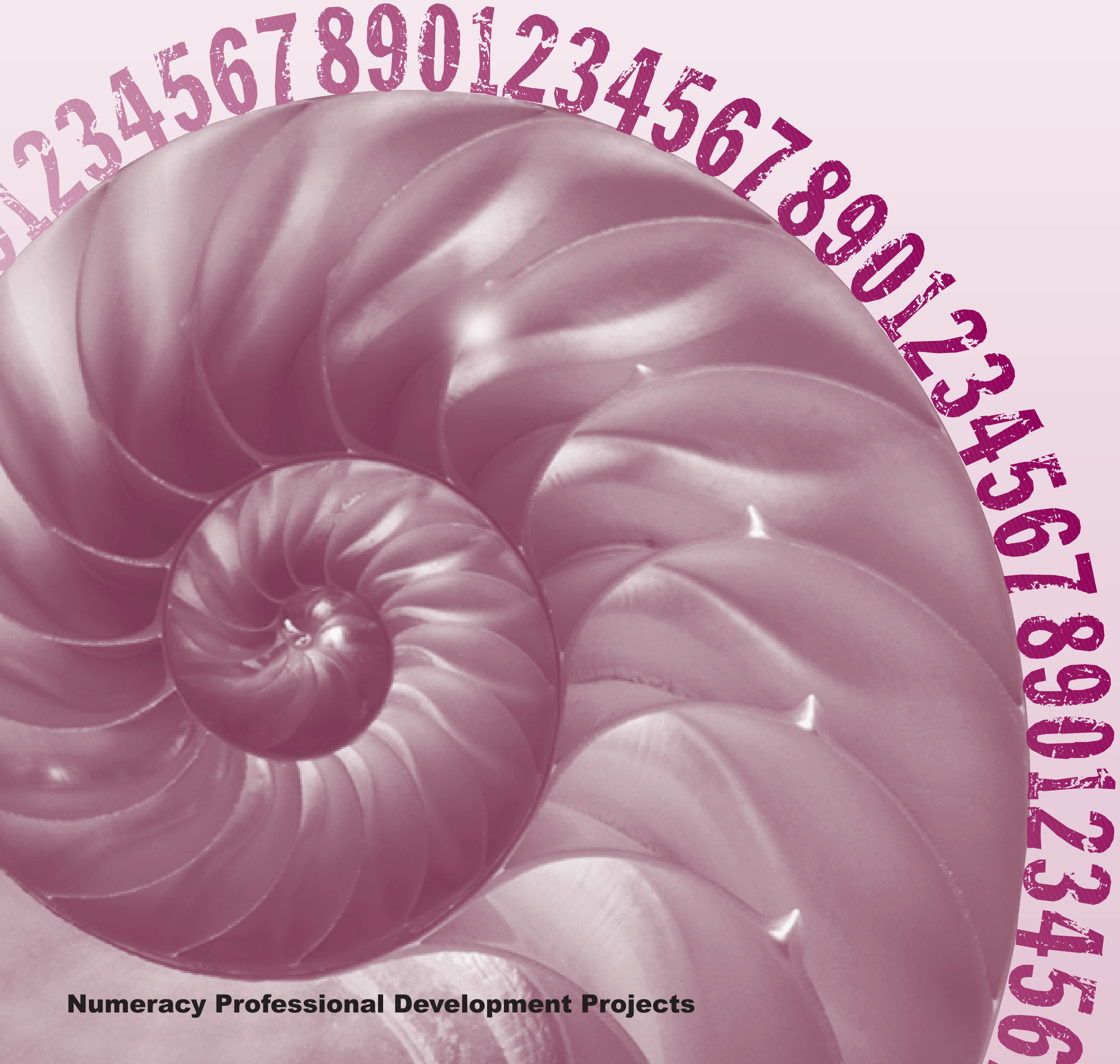
MINISTRY OF EDUCATION
Te Tāhuhu o te Mātauranga

Book 1
The Number Framework
Revised Edition 2007



Numeracy Professional Development Projects

THE NUMERACY DEVELOPMENT PROJECTS

Teachers are key figures in changing the way in which mathematics and statistics is taught and learned in schools. Their subject matter and pedagogical knowledge are critical factors in the teaching of mathematics and statistics understanding. The effective teacher of mathematics and statistics has a thorough and deep understanding of the subject matter to be taught, how students are likely to learn it, and the difficulties and misunderstandings they are likely to encounter.

The focus of the Numeracy Development Projects is to improve student performance in mathematics through improving the professional capability of teachers. To date, almost every teacher of year 1 to 6 children and the majority of teachers of year 7 and 8 children have had the opportunity to participate.

A key feature of the projects is their dynamic and evolutionary approach to implementation. This ensures that the projects can be informed by developing understanding about mathematics learning and effective professional development and that flexibility in approach and sector involvement is maximised.

The projects continue to build on the findings and experience associated with the numeracy professional development projects that operated in 2002–2007. These projects made an important contribution to what we know about:

- children's learning and thinking strategies in early mathematics;
- effective identification of, and response to, children's learning needs;
- the characteristics of professional development programmes that change teaching practice; and
- effective facilitation.

Such findings continue to inform the modification and further development of the projects. National co-ordinators and facilitators from each region provide ongoing feedback about aspects of the projects.

Note: *Book 1: The Number Framework* was modified in 2007 in response to research findings and facilitator feedback. A paragraph on calculation methods (see page 14) was added in 2008.

Numeracy Professional Development Projects 2008
Published by the Ministry of Education.
PO Box 1666, Wellington 6140, New Zealand.

ISBN 978 0 7903 1668 0
Dewey number 372.7
Topic Dewey number 510
Item number 31668

Note: Teachers may copy these notes for educational purposes.

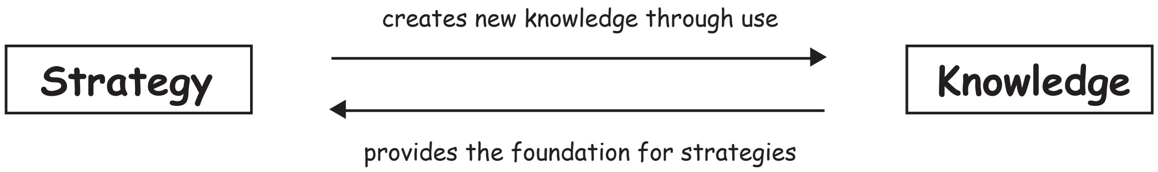
This book is also available on the New Zealand Maths website, at www.nzmaths.co.nz

The Number Framework

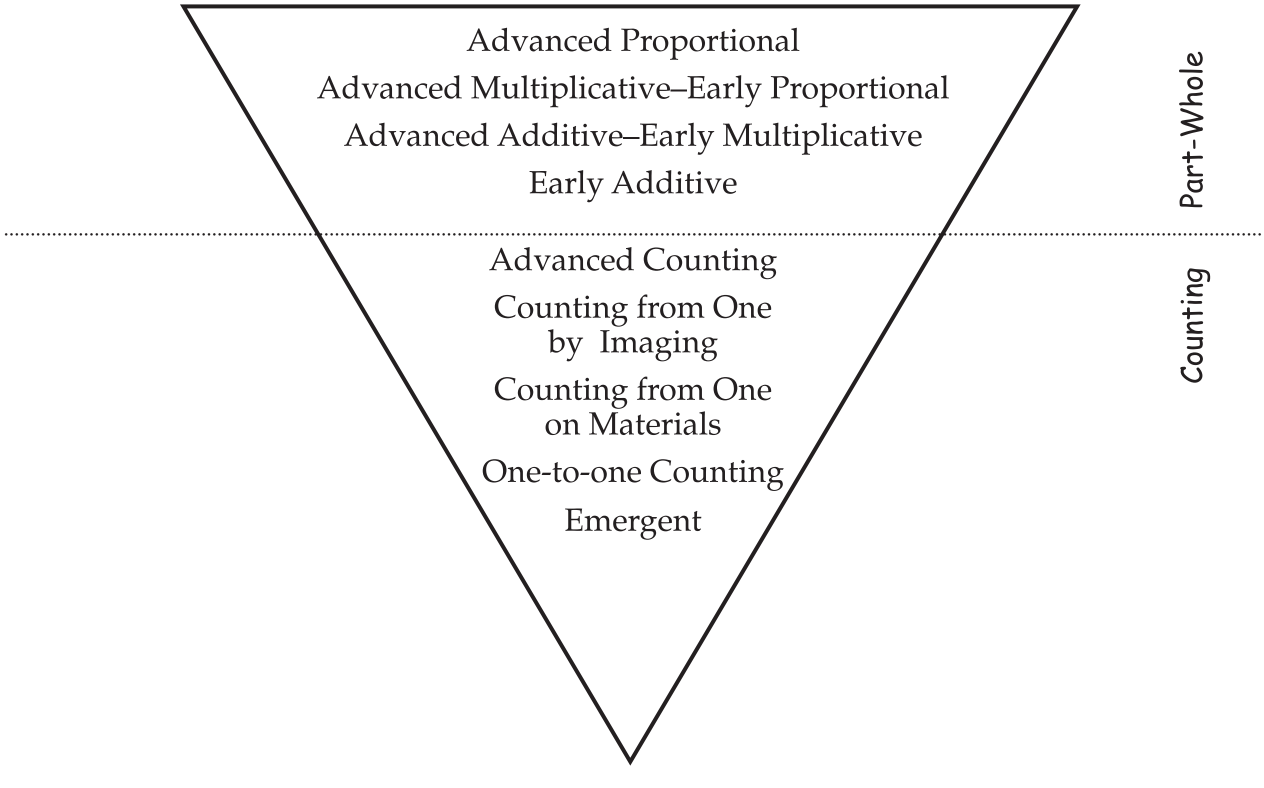
Introduction

The Number Framework helps teachers, parents, and students to understand the requirements of the Number and Algebra strand of the mathematics and statistics curriculum learning area. The Framework embodies most of the achievement aims and objectives in levels 1 to 4. In the two main sections of the Framework, the distinction is made between strategy and knowledge. The *strategy* section describes the mental processes students use to estimate answers and solve operational problems with numbers. The *knowledge* section describes the key items of knowledge that students need to learn.

It's important that students make progress in both sections of the Framework. Strong knowledge is essential for students to broaden their strategies across a full range of numbers, and knowledge is often an essential prerequisite for the development of more advanced strategies. For example, a student is unlikely to solve 9 + 6 as 10 + 5 if he or she does not know the "ten and" structure of teen numbers. Similarly, using more advanced strategies helps students to develop knowledge. For example, a student who uses doubling of the multiplication by two facts to work out the multiplication by four facts is likely to learn these facts through appropriate repetition.



The strategy section of the Framework consists of a sequence of global stages. The diagram below presents the strategy stages as an inverted triangle. Progress through the stages indicates an expansion in knowledge and in the range of strategies that students have available. The triangle also suggests that students build new strategies on their existing strategies and that these existing strategies are not subsumed. Students frequently revert to previous strategies when presented with unfamiliar problems, or when the mental load gets high.



The global stages make it easier to identify and describe the types of mental strategies used by students. An important aim of teaching is helping students to develop more sophisticated strategies. Students appear to be very consistent in their approach to operating with numbers. Their consistency will help you to anticipate what strategies students are likely to use across a broad range of problems and to plan appropriate learning activities and questions.

An Overview of the Strategy Section of the Framework

Each strategy stage is described below. Links are made to the views that students hold about numbers and how numbers can be manipulated to solve operational problems. Experience with using the *Numeracy Project Assessment (NumPA)* will help you to understand these stages.

The table below gives the structure of the strategy section of the Framework. Each stage contains the operational domains of addition and subtraction, multiplication and division, and proportions and ratios.

			Addition Subtraction	Multiplication Division	Proportions Ratios
Counting	Zero	Emergent			
	One	One-to-one Counting			
	Two	Counting from One on Materials			
	Three	Counting from One by Imaging			
	Four	Advanced Counting			
Part-Whole	Five	Early Additive Part-Whole			
	Six	Advanced Additive–Early Multiplicative Part-Whole			
	Seven	Advanced Multiplicative–Early Proportional Part-Whole			
	Eight	Advanced Proportional Part-Whole			

Students are sometimes between stages. That is, they display characteristics of one stage given a certain problem but use a more-or-less advanced strategy given a different problem. Gaps in key knowledge are usually the reason. For example, a student may use doubles to solve 8 + 7 as 7 + 7 + 1 but count on to work out 9 + 5 as the teen number code is not known.

A close stage match in progress across the three domains is most common and should be seen as a desirable growth path for students. However, for some students, their progress across the operational domains is not in phase. For example, on NumPA, some students understand how to derive multiplication answers from known facts but do so by counting on or back. They are unlikely to derive multiplication answers independently in this way. For addition and subtraction, they are rated at the Advanced Counting stage, but they are rated at the Advanced Additive–Early Multiplicative stage for multiplication and division.

Strategy Stages

This section contains descriptions of the strategies that students use at each stage. Some specific examples are given to illustrate student thinking. There are many other examples of strategies that students may use at each stage.

Stage Zero: Emergent

Students at this stage are unable to consistently count a given number of objects because they lack knowledge of counting sequences and/or the ability to match things in one-to-one correspondence.

Stage One: One-to-one Counting

This stage is characterised by students who can count and form a set of objects up to ten but cannot solve simple problems that involve joining and separating sets, like 4 + 3.

Stage Two: Counting from One on Materials

Given a joining or separating of sets problem, students at this stage rely on counting physical materials, like their fingers. They count all the objects in both sets to find an answer, as in "Five lollies and three more lollies. How many lollies is that altogether?"

Stage Three: Counting from One by Imaging

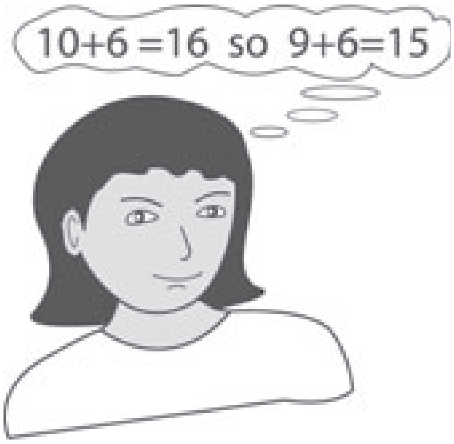
This stage is also characterised by students counting all of the objects in simple joining and separating problems. Students at this stage are able to image visual patterns of the objects in their mind and count them.

Stage Four: Advanced Counting (Counting On)

Students at this stage understand that the end number in a counting sequence measures the whole set and can relate the addition or subtraction of objects to the forward and backward number sequences by ones, tens, etc. For example, instead of counting all objects to solve 6 + 5, the student recognises that "6" represents all six objects and counts on from there: "7 , 8 , 9, 10, 11."

Students at this stage also have the ability to co-ordinate equivalent counts, such as “10, 20, 30, 40, 50,” to get $50 in $10 notes. This is the beginning of grouping to solve multiplication and division problems.

Stage Five: Early Additive Part-Whole



At this stage, students have begun to recognise that numbers are abstract units that can be treated simultaneously as wholes or can be partitioned and recombined. This is called *part-whole thinking.*

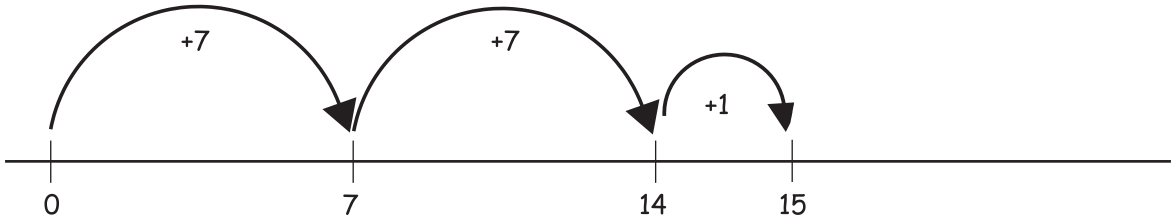
A characteristic of this stage is the derivation of results from related known facts, such as finding addition answers by using doubles or teen numbers.

The strategies that these students commonly use can be represented in various ways, such as empty number lines, number strips, arrays, or ratio tables.

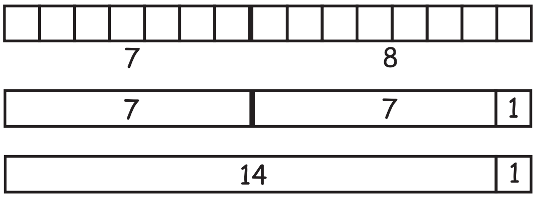
For example:

(i) Compensation from known facts
Example: 7 + 8: 7 + 7 is 14, so 7 + 8 is 15.

Number line:

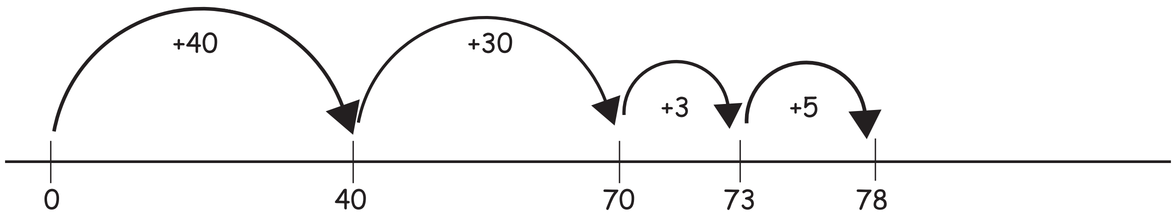


Number strip:

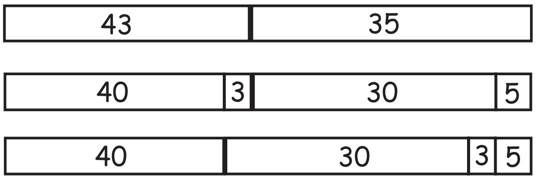


(ii) Standard place value partitioning
Example: 43 + 35 is (40 + 30) + (3 + 5) = 70 + 8.

Number line:



Number strip:



Stage Six: Advanced Additive–Early Multiplicative Part-Whole

Students at this stage are learning to choose appropriately from a repertoire of part-whole strategies to solve and estimate the answers to addition and subtraction problems. They see numbers as whole units in themselves but also understand that "nested" within these units is a range of possibilities for subdivision and recombining.

Simultaneously, the efficiency of these students in addition and subtraction is reflected in their ability to derive multiplication answers from known facts. These students can also solve fraction problems using a combination of multiplication and addition-based reasoning.

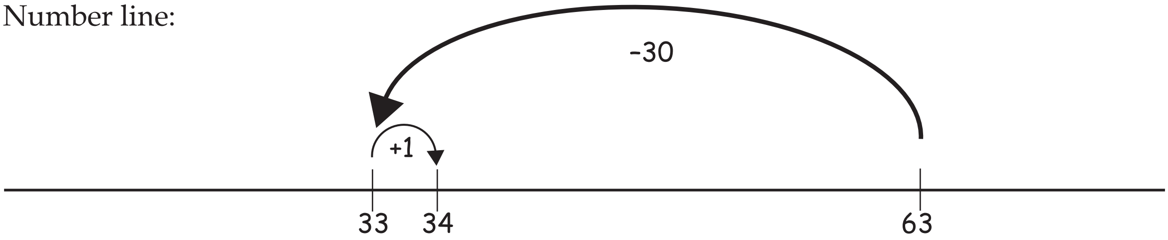
For example, 6×6 as $(5 \times 6) + 6$;

or $\frac{3}{4}$ of 24 as $\frac{1}{4}$ of 20 is 5 because $4 \times 5 = 20$, so $\frac{3}{4}$ of 20 is 15, and $\frac{1}{4}$ of 4 is 1 because $4 \times 1 = 4$, so $\frac{3}{4}$ of 4 is 3. Therefore $\frac{3}{4}$ of 24 is 15 + 3, namely 18.

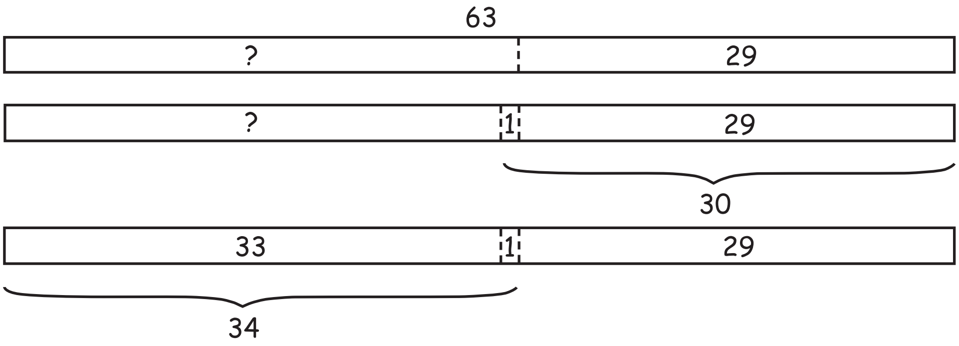
Here are some examples of the addition and subtraction strategies used by Advanced Additive Part-Whole students:

(i) Standard place value with tidy numbers and compensation
Example: $63 - 29 = \square$ as $63 - 30 + 1 = \square$.

Number line:



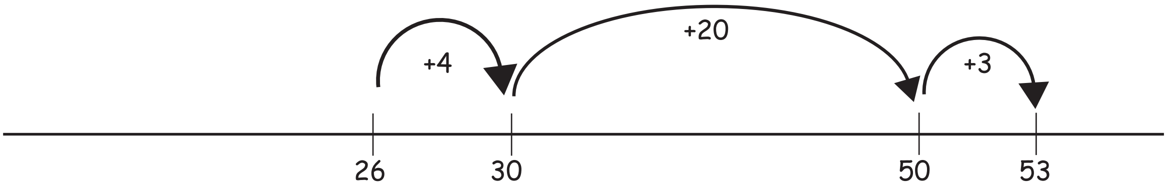
Number strip:



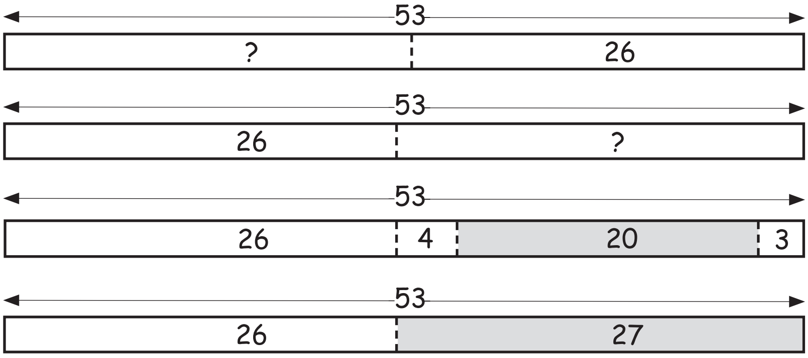
(ii) Reversibility

Example: 53 – 26 = ☐ as 26 + ☐ = 53. 26 + (4 + 20 + 3) = 53, so 53 – 26 = 27.

Number line:



Number Strip:

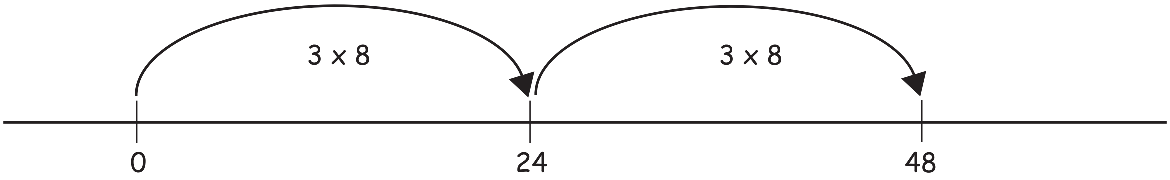


Advanced Additive students also use addition strategies to derive multiplication facts. Their strategies usually involve partitioning factors additively. Here are two examples of such strategies:

(i) Doubling

Example: 3 × 8 = 24, so 6 × 8 = 24 + 24 = 48.

Number line:



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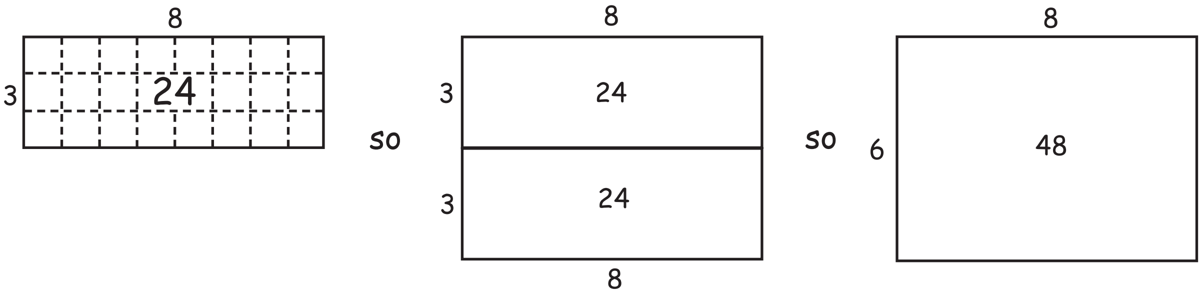
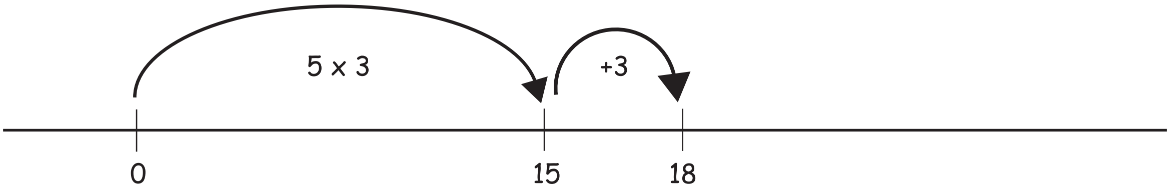


(ii) Compensation

Example: 5 × 3 = 15, so 6 × 3 = 18 (three more; compensation using addition)

Number line:



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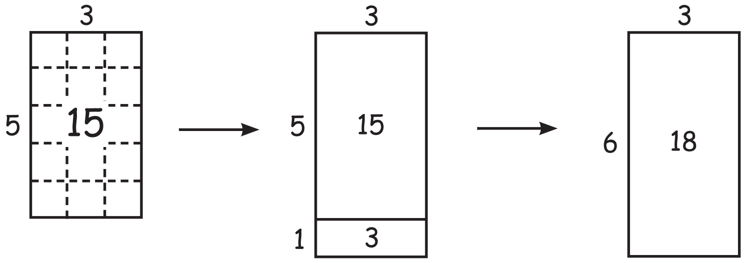
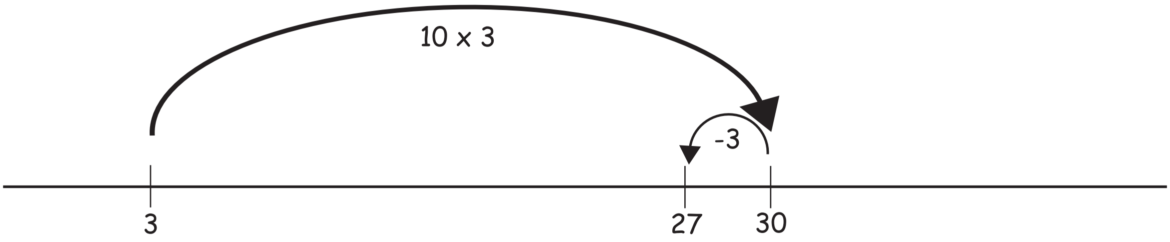


Example: 10 × 3 = 30, so 9 × 3 = 27 (three less; compensation using subtraction)

Number line:



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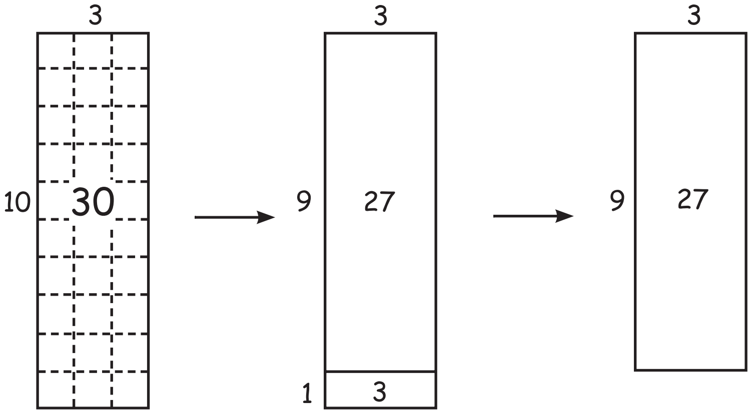


Stage Seven: Advanced Multiplicative–Early Proportional Part-Whole

Students at this stage are learning to choose appropriately from a range of part-whole strategies to solve and estimate the answers to problems involving multiplication and division. These strategies require one or more of the numbers involved in a multiplication or division to be partitioned, manipulated, then recombined.

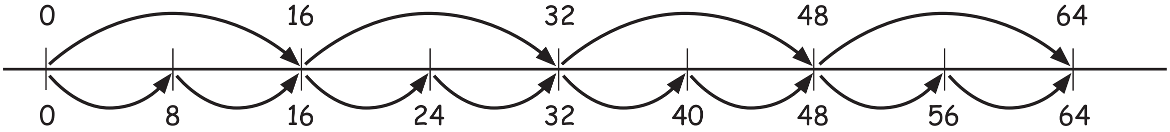
For example, to solve 27 × 6, 27 might be split into 20 + 7 and these parts multiplied then recombined, as in 20 × 6 = 120, 7 × 6 = 42, 120 + 42 = 162, or 2 × 27 = 54, 3 × 54 = 162. The first strategy partitions 27 additively, the second strategy partitions 6 multiplicatively.

A critical development at this stage is the use of reversibility, in particular, solving division problems using multiplication. Advanced Multiplicative Part-Whole students are also able to solve and estimate the answers to problems with fractions using multiplication and division. For example, to solve $\frac{2}{3}$ of ☐ = 18, $\frac{1}{2}$ of 18 = 9, ☐ = 3 × 9 = 27 (using unit fractions). Students, at this stage, can also understand the multiplicative relationship between the numerators and denominators of equivalent fractions, e.g. $\frac{3}{4} = \frac{75}{100}$.

Here are some strategies used by students at the Advanced Multiplicative stage to solve multiplication and division problems:

(i) Doubling and halving or trebling and dividing by three (thirding)
Example: 4 × 16 as 8 × 8 = 64 (doubling and halving)

Double number line:



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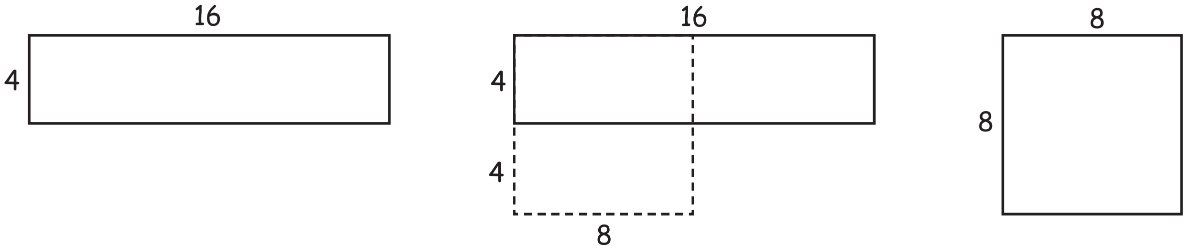
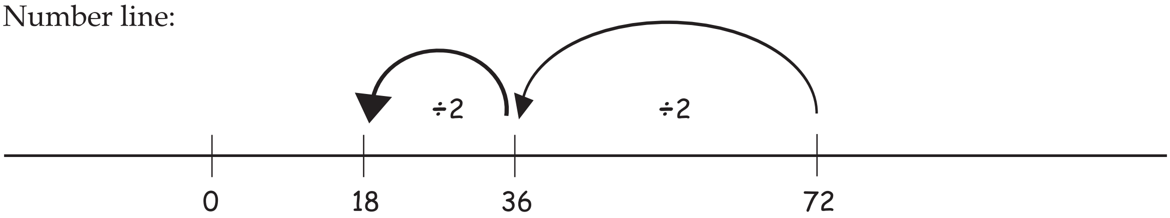


(ii) Halving and halving
Example: 72 ÷ 4 as 72 ÷ 2 = 36, 36 ÷ 2 = 18 (dividing by four is the same as dividing by two twice).

Number line:



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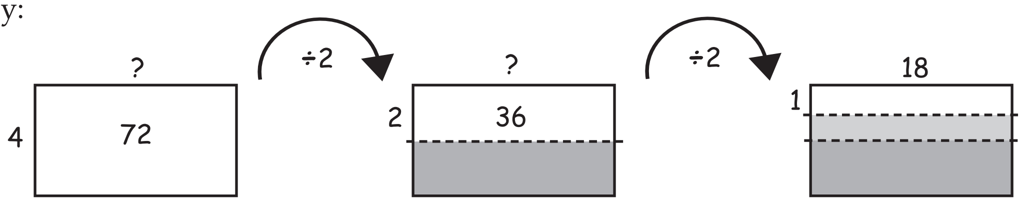
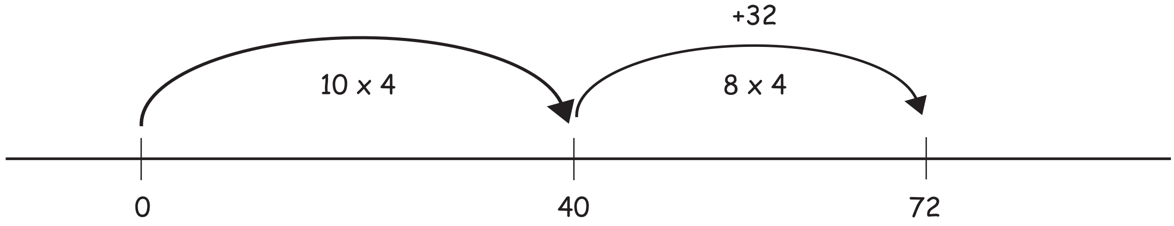


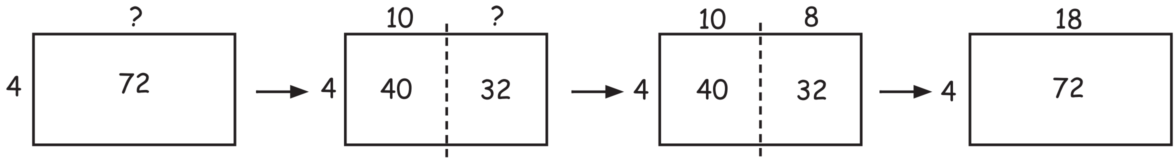
(Not drawn to scale)

(iii) Reversibility and place value partitioning
Example: 72 ÷ 4 as 10 × 4 = 40, 72 – 40 = 32, 8 × 4 = 32, 10 + 8 = 18, so 18 × 4 = 72

Number line:



Array (reversibility)



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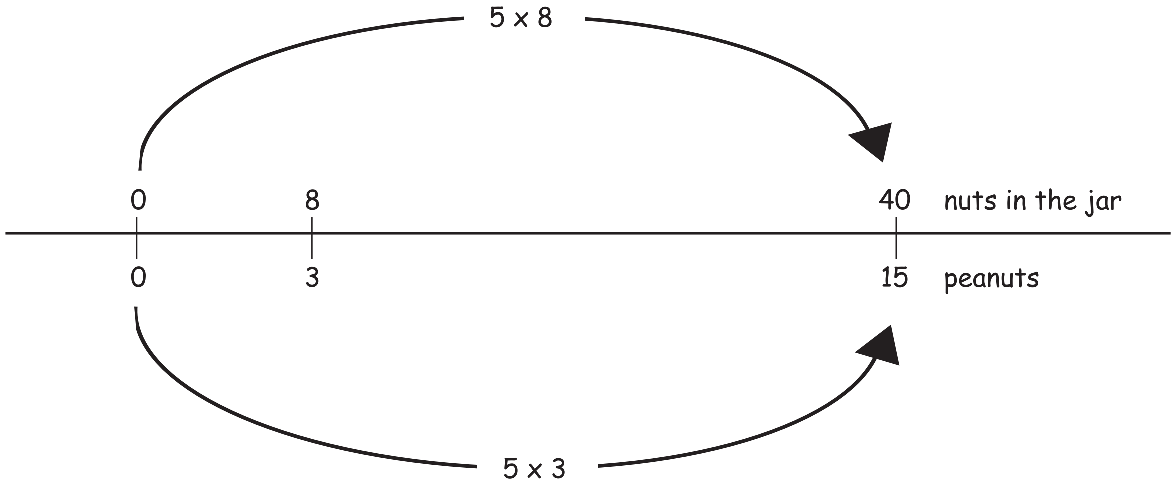
Advanced Multiplicative students also apply their strengths in multiplication and division to problems involving fractions, proportions and ratios. Generally, their strategies involve using equivalent fractions. Here are some examples of the strategies they use:

(i) Multiplying within
Example: Every packet in the jar has 8 nuts in it. Three of the 8 nuts are peanuts. The jar contains 40 nuts altogether, all in packets. How many of the nuts are peanuts?
(3:8 as ☐:40, 5 × 8 = 40 so 5 × 3 = ☐)

Double number line:

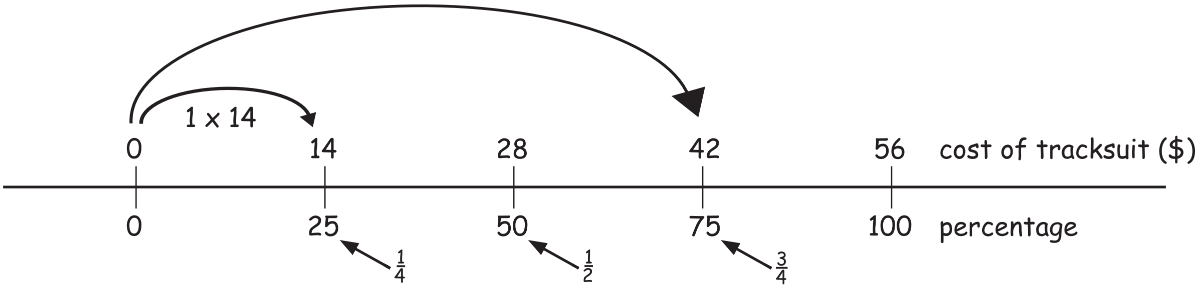


(ii) Using unit fractions and conversion from percentages

Example: The tracksuit is usually \$56, but the shop has a 25% off sale. How much does Mere pay?

25% is $\frac{1}{4}$, so Mere pays $\frac{3}{4}$ of \$56. $\frac{1}{4}$ of 56 is $56 \div 4 = 14$. $\frac{3}{4} = 3 \times \frac{1}{4}$, $3 \times 14 = 42$

Double number line:



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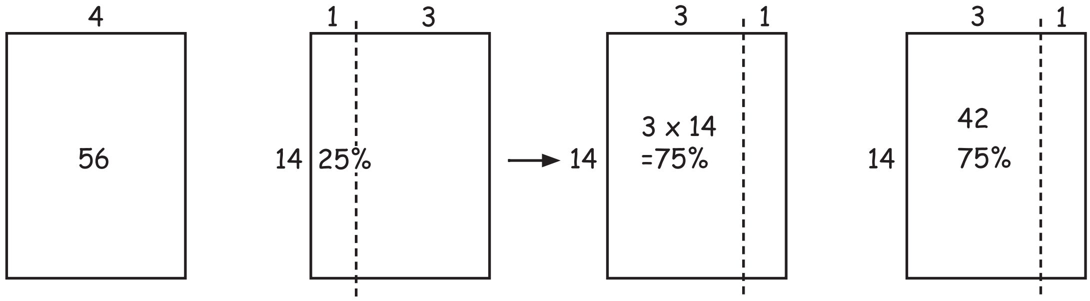


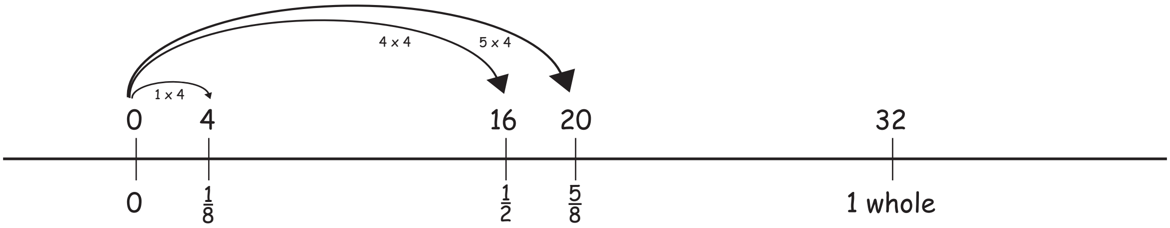
(Not drawn to scale)

(ii) Using the distributive property

Example: Albert has 32 matchbox toys. Five-eighths of them are sports cars. How many of the matchbox toys are sports cars?

Method 1: $\frac{5}{8}$ of 32 as $\frac{1}{2}$ of 32 is 16 and $\frac{1}{8}$ of 32 is 4, so $\frac{5}{8}$ of 32 is 16 + 4 = 20

Double number line:



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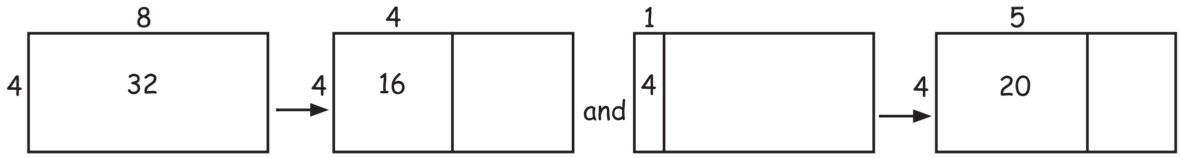
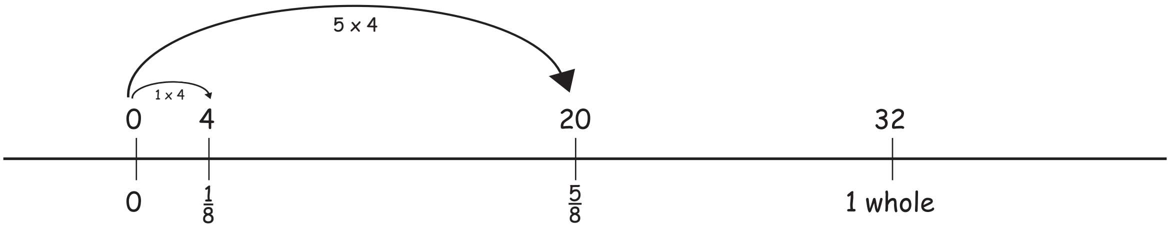


Method 2: $\frac{1}{8}$ of 32 is 4, so $\frac{5}{8}$ of 32 is $5 \times 4 = 20$

Double number line:



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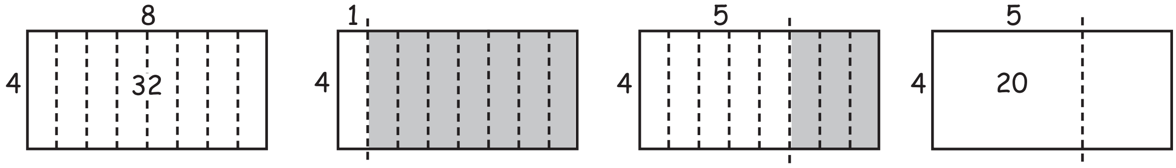


Stage Eight: Advanced Proportional Part-Whole

Students at this stage are learning to select from a repertoire of part-whole strategies to solve and estimate the answers to problems involving fractions, proportions, and ratios. These strategies are based on finding common factors and include strategies for the multiplication of decimals and the calculation of percentages.

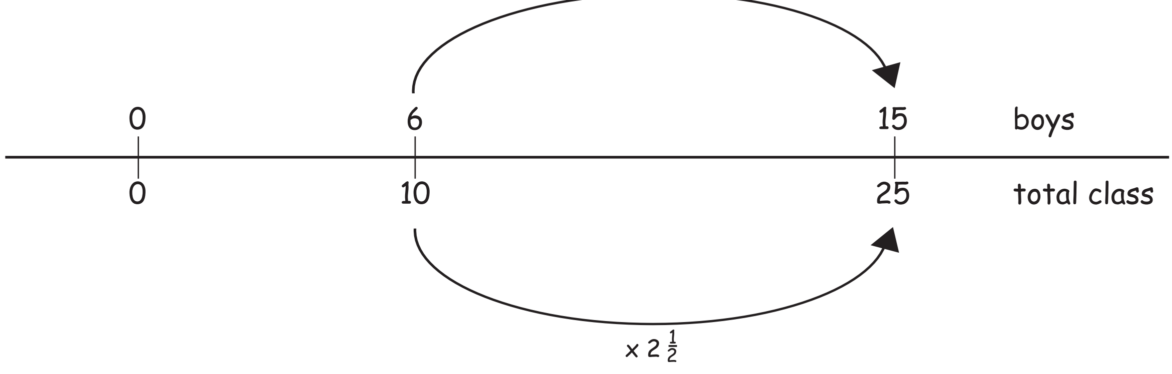
These students are able to find the multiplicative relationship between quantities of two different measures. This can be thought of as a mapping. For example, consider this problem: "You can make 21 glasses of lemonade from 28 lemons. How many glasses can you make using 8 lemons?"

To solve the problem, students may need to find a relationship between the number of lemons and the number of glasses. This involves the creation of a new measure, glasses per lemon. The relationship is that the number of glasses is three-quarters the number of lemons. This could be recorded as: 21:28 is equivalent to □ :8? 21 is $\frac{3}{4}$ of 28 $\Rightarrow$ 6 is $\frac{3}{4}$ of 8.

Double number line:

Some problems can be solved by finding the relationship within units.
Example: Of every 10 children in the class, 6 are boys. There are 25 children in the class. How many of them are boys?

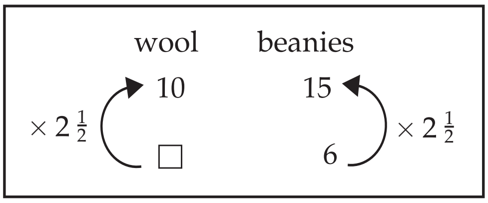
Double number line:



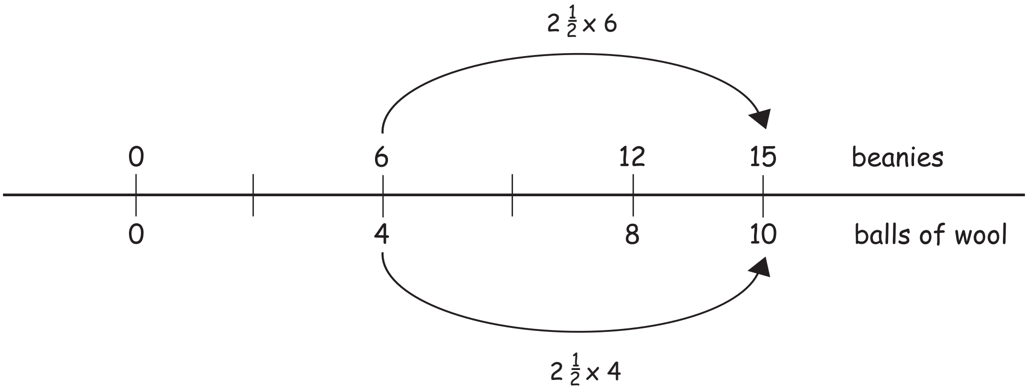
Advanced Proportional students may use either within or between measures as strategies to solve a fractional multiplication problem.

Example: It takes 10 balls of wool to make 15 beanies. How many balls of wool make 6 beanies?

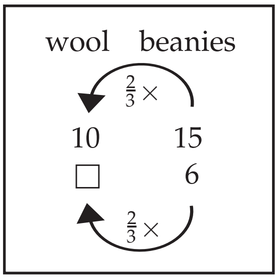
Within measures: $2\frac{1}{2} \times 6 = 15$, so $2\frac{1}{2} \times \square = 10$.



Double number line:



Between measures: $\frac{2}{3} \times 15 = 10$, so $\frac{2}{3} \times 6 = 4$.



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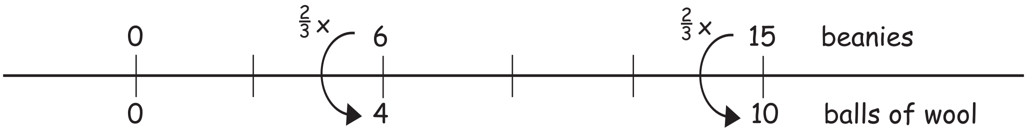


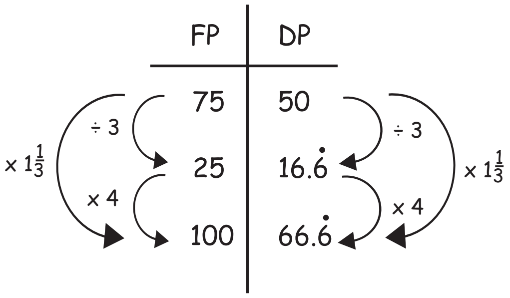
Example: Cayla's Clothing Shop is giving a discount. For a \$75 pair of jeans, you only pay \$50. What percentage discount is that?

The discount is \$75 – \$50 = \$25. As a fraction of the original price of \$75, this discount is $\frac{1}{3}$. So the percentage discount is $33.\dot{3}\%$

Ratio tables:

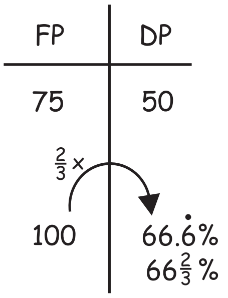
Full price	Discount price
75	50
100	?

Within strategy

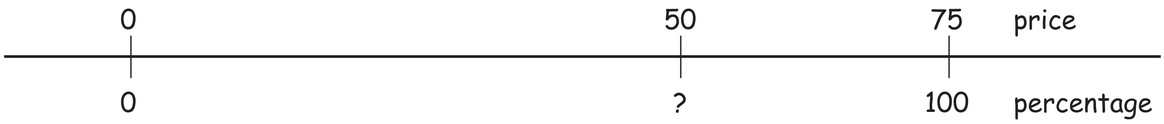


or

Between strategy



Number lines:



An Overview of the Knowledge Section of the Framework

The knowledge section of the Framework outlines the important items of knowledge that students should learn as they progress through the strategy stages. This knowledge plays a critical role in students applying their available strategies with proficiency and fluency across all the numbers and problem types that they may encounter.

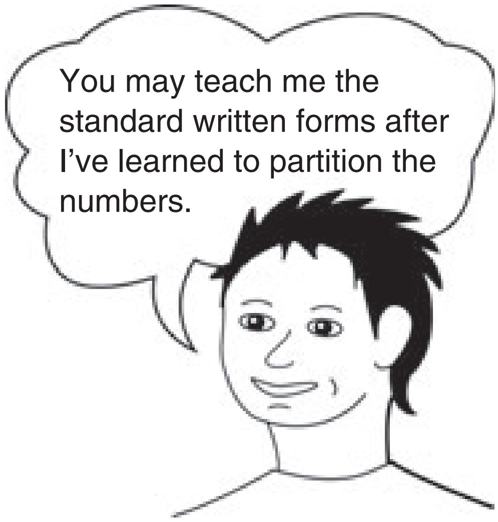
It is important to avoid a "culture of limits" when exposing students to the teaching of number knowledge. The same items of knowledge can be taught to students across a range of strategy stages. For example, students can come to read and write decimals long before their strategies are sufficiently developed for them to use decimals in operational problems.

In the Framework, knowledge is categorised under four content domains: Number Identification, Number Sequence and Order, Grouping/Place Value, and Basic Facts. Progressions for written recording are indicated on pages 18–22 in a fifth column.

Written recording can be seen as a thinking tool, a communication tool, and a reflective tool. It is critical that, at every stage, students engage in building meaning through recording their mathematical ideas in the form of pictures, diagrams, words, and symbols. Making explicit the links between oral and written forms is fundamental to the sound development of the meaning and structure of the language of mathematics. This oral and written language underpins all understanding. The informal jottings of students are to be encouraged as a way to capture their mental processes so that their ideas can be shared with others. **Students should not be exposed to standard written algorithms until they use part-whole mental strategies.** Premature exposure to working forms restricts students' ability and desire to use mental strategies. This inhibits their development of number sense. However, in time, written methods must become part of a student's calculation repertoire.

Calculators and computers can also allow students to find answers to calculations that may be beyond their mental and written capacity. Access to this technology allows students to model real world problems in which the numbers may be otherwise prohibitive, to explore patterns in number relationships, and to concentrate on conceptual understanding without the cognitive load of calculation. In using such tools, other mathematical thinking is required, including evaluating the validity of the mathematical model chosen for solving the problem and checking the "reasonableness" of the answer produced.

Counting needs to be developed with all students. This counting should extend past counting whole numbers by ones, twos, and other simple multiples into recognising patterns in powers of ten, in decimals, and in fractions. The Framework also acknowledges the importance of backwards counting sequences, particularly in the early years.



The Framework encourages greater use of students' natural inclination to use groupings of fives. There is significant emphasis on using finger patterns in the early years and in using multiples of five in later work, for example, 35 + 35 = 70, 500 + 500 = 1000.

Basic fact knowledge is critical. The Number Framework emphasises that the process of deriving number facts using mental strategies is important in coming to know and apply these facts. It also demands that students come to know a broader range of facts than previously, including groupings of "benchmark" numbers, and that they have knowledge of factors of numbers and decimal and fraction conversions at the higher stages.

The Number Framework – Strategies

		Operational Domains		
	Global Stage	**Addition and Subtraction**	**Multiplication and Division**	**Proportions and Ratios**
Counting	**Zero: Emergent**	**Emergent** The student is unable to count a given set or form a set of up to ten objects.		
	One: One-to-one Counting	**One-to-one Counting** The student is able to count a set of objects but is unable to form sets of objects to solve simple addition and subtraction problems.	**One-to-one Counting** The student is able to count a set of objects but is unable to form sets of objects to solve simple multiplication and division problems.	**Unequal Sharing** The student is unable to divide a region or set into two or four equal parts.
	Two: Counting from One on Materials	**Counting – from One** The student solves simple addition and subtraction problems by counting all the objects, e.g., 5 + 4 as 1, 2, 3, 4, 5, 6, 7, 8, 9. The student needs supporting **materials**, like fingers.	**Counting – from One** The student solves multiplication and division problems by counting one to one with the aid of **materials**.	**Equal Sharing** The student is able to divide a region or set into given equal parts using **materials**. With sets this is done by equal sharing of materials. With shapes symmetry (halving) is used.
	Three: Counting from One by Imaging	**Counting – from One** The student **images** all of the objects and counts them. The student does not see ten as a unit of any kind and solves multi-digit addition and subtraction problems by counting all the objects.	**Counting – from One** The student **images** the objects to solve simple multiplication and division problems, by counting all the objects, e.g., 4 × 2 as 1, **2**, 3, **4**, 5, **6**, 7, **8**. For problems involving larger numbers the student will still rely on materials.	**Equal Sharing** The student is able to share a region or set into given equal parts by using **materials** or by **imaging** the materials for simple problems, e.g., $\frac{1}{2}$ of 8. With sets this is done by equal sharing of materials or by imaging. With shapes symmetry is used to create halves, quarters, eighths, etc.
	Four: Advanced Counting	**Counting On** The student uses counting on or counting back to solve simple addition or subtraction tasks, e.g., 8 + 5 by 8, 9, 10, 11, 12, 13 or 52 – 4 as 52, 51, 50, 49, 48. Initially, the student needs supporting **materials** but later **images** the objects and counts them. The student sees 10 as a completed count composed of 10 ones. The student solves addition and subtraction tasks by incrementing in ones (38, 39, 40, ...), tens counts (13, 23, 33, ...), and/or a combination of tens and ones counts (27, 37, 47, 48, 49, 50, 51).	**Skip-counting** On multiplication tasks, the student uses skip-counting (often in conjunction with one-to-one counting), e.g., 4 × 5 as 5, 10, 15, 20. The student may track the counts using materials (eg. fingers) or by imaging.	

		Operational Domains		
Global Stage		Addition and Subtraction	Multiplication and Division	Proportions and Ratios
Part-Whole	**Five: Early Additive Part-Whole**	**Early Addition and Subtraction** The student uses a limited range of mental strategies to estimate answers and solve addition or subtraction problems. These strategies involve deriving the answer from known basic facts, e.g., 8 + 7 is 8 + 8 – 1 (doubles) or 5 + 3 + 5 + 2 (fives) or 10 + 5 (making tens). Their strategies with multi-digit numbers involve using tens and hundreds as abstract units that can be partitioned, e.g., 43 + 25 = (40 + 20) + (3 + 5) = 60 + 8 = 68 (standard partitioning) or 39 + 26 = 40 + 25 = 65 (rounding and compensation) or 84 – 8 as 84 – 4 – 4 = 76 (back through ten).	**Multiplication by Repeated Addition** On multiplication tasks, the student uses a combination of known multiplication facts and repeated addition, e.g., 4 × 6 as (6 + 6) + (6 + 6) = 12 + 12 = 24. The student uses known multiplication and repeated addition facts to anticipate the result of division, e.g., 20 ÷ 4 = 5 because 5 + 5 = 10 and 10 + 10 = 20.	**Fraction of a Number by Addition** The student finds a fraction of a number and solves division problems with remainders mentally using halving, or deriving from known addition facts, e.g., $\frac{1}{3}$ of 12 is 4 because 3 + 3 + 3 = 9, so 4 + 4 + 4 = 12; e.g., 7 pies shared among 4 people (7 ÷ 4) by giving each person 1 pie, and $\frac{1}{2}$ pie, then $\frac{1}{4}$ pie.
	Six: Advanced Additive (Early Multiplicative) Part-Whole	**Advanced Addition and Subtraction of Whole Numbers** The student can estimate answers and solve addition and subtraction tasks involving whole numbers mentally by choosing appropriately from a broad range of advanced mental strategies, e.g., 63 – 39 = 63 – 40 + 1 = 24 (rounding and compensating) or 39 + 20 + 4 = 63, so 63 – 39 = 24 (reversibility) or 64 – 40 = 24 (equal additions) e.g., 324 – 86 = 300 – 62 = 238 (standard place value partitioning) or 324 – 100 + 14 = 238 (rounding and compensating).	**Derived Multiplication** The student uses a combination of known facts and mental strategies to derive answers to multiplication and division problems, e.g., 4 × 8 = 2 × 16 = 32 (doubling and halving), e.g., 9 × 6 is (10 × 6) – 6 = 54 (rounding and compensating), e.g., 63 ÷ 7 = 9 because 9 × 7 = 63 (reversibility).	**Fraction of a Number by Addition and Multiplication** The student uses repeated halving or known multiplication and division facts to solve problems that involve finding fractions of a set or region, renaming improper fractions, and division with remainders, e.g., $\frac{1}{3}$ of 36, 3 × 10 = 30, 36 – 30 = 6, 6 ÷ 3 = 2, 10 + 2 = 12 e.g., $\frac{16}{3} = 5\frac{1}{3}$ (using 5 × 3 = 15) e.g., 8 pies shared among 3 people (8 ÷ 3) by giving each person 2 pies and dividing the remaining 2 pies into thirds (answer: $2 + \frac{1}{3} + \frac{1}{3} = 2\frac{2}{3}$). The student uses repeated replication to solve simple problems involving ratios and rates, e.g. 2:3 → 4:6 → 8:12 etc.

		Operational Domains		
	Global Stage	**Addition and Subtraction**	**Multiplication and Division**	**Proportions and Ratios**
Part-Whole	**Seven: Advanced Multiplicative (Early Proportional) Part-Whole**	**Addition and Subtraction of Decimals and Integers** The student can choose appropriately from a broad range of mental strategies to estimate answers and solve addition and subtraction problems involving decimals, integers, and related fractions. The student can also use multiplication and division to solve addition and subtraction problems with whole numbers. e.g., $3.2 + 1.95 = 3.2 + 2 - 0.05$ $= 5.2 - 0.05$ $= 5.15$ (compensation); e.g., $6.03 - 5.8 = \square$ as $6.03 - 5 - 0.8 = 1.03 - 0.8 = 0.23$ (standard place value partitioning) or as $5.8 + \square = 6.03$ (reversibility) e.g., $\square + 3.98 = 7.04$ as $3.98 + \square = 7.04$, $\square = 0.02 + 3.04 = 3.06$ (commutativity) e.g., $\frac{3}{4} + \frac{5}{8} = (\frac{3}{4} + \frac{2}{8}) + \frac{3}{8} = 1\frac{3}{8}$ (partitioning fractions) e.g., $81 - 36 = (9 \times 9) - (4 \times 9) = 5 \times 9$ (using factors) e.g., $28 + 33 + 27 + 30 + 32 = 5 \times 30$ (averaging) e.g., $^+7 - {}^-3 = {}^+7 + {}^+3 = {}^+10$ (equivalent operations on integers)	**Advanced Multiplication and Division** The student chooses appropriately from a broad range of mental strategies to estimate answers and solve multiplication and division problems. These strategies involve partitioning one or more of the factors in multiplication, and applying reversibility to solve division problems, particularly those involving missing factors and remainders. The partitioning may be additive or multiplicative. e.g., $24 \times 6 = (20 \times 6) + (4 \times 6)$ (place value partitioning) or $25 \times 6 - 6$ (rounding and compensating) e.g., $81 \div 9 = 9$, so $81 \div 3 = 3 \times 9$ (proportional adjustment) e.g., $4 \times 25 = 100$, so $92 \div 4 = 25 - 2 = 23$ (reversibility and rounding with compensation) e.g., $90 \div 5 = 18$ so $87 \div 5 = 17$ r 2 (rounding and divisibility) e.g., $201 \div 3$ as $100 \div 3 = 33$ r 1, $200 \div 3 = 66$ r 2, $201 \div 3 = 67$ (divisibility rules)	**Fractions, Ratios, and Proportions by Multiplication** The student uses a range of multiplication and division strategies to estimate answers and solve problems with fractions, proportions, and ratios. These strategies involve linking division to fractional answers, e.g., $11 \div 3 = \frac{11}{3} = 3\frac{2}{3}$ e.g., $13 \div 5 = (10 \div 5) + (3 \div 5) = 2\frac{3}{5}$ The student can also find simple equivalent fractions and rename common fractions as decimals and percentages. e.g., $\frac{5}{6}$ of 24 as $\frac{1}{6}$ of $24 = 4$, $5 \times 4 = 20$ or $24 - 4 = 20$ e.g., 3:5 as $\square$: 40, $8 \times 5 = 40$, $8 \times 3 = 24$ so $\square = 24$. e.g., $\frac{3}{4} = \frac{75}{100} = 75\% = 0.75$
	Eight: Advanced Proportional Part-Whole	**Addition and Subtraction of Fractions** The student uses a range of mental partitioning strategies to estimate answers and solve problems that involve adding and subtracting fractions, including decimals. The student is able to combine ratios and proportions with different amounts. The strategies include using partitions of fractions and "ones", and finding equivalent fractions. e.g., $2\frac{3}{4} - 1\frac{2}{3} = 1 + (\frac{3}{4} - \frac{2}{3}) = 1 + (\frac{9}{12} - \frac{8}{12}) = 1\frac{1}{12}$ (equivalent fractions) e.g., 20 counters in ratio of 2:3 combined with 60 counters in ratio 8:7 gives a combined ratio of 1:1.	**Multiplication and Division of Decimals/ Multiplication of Fractions** The student chooses appropriately from a range of mental strategies to estimate answers and solve problems that involve the multiplication of fractions and decimals. The student can also use mental strategies to solve simple division problems with decimals. These strategies involve the partitioning of fractions and relating the parts to one, converting decimals to fractions and vice versa, and recognising the effect of number size on the answer, e.g., $3.6 \times 0.75 = \frac{3}{4} \times 3.6 = 2.7$ (conversion and commutativity); e.g., $\frac{2}{3} \times \frac{3}{4} = \square$ as $\frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$ so $\frac{2}{3} \times \frac{1}{4} = \frac{2}{12}$ so $\frac{2}{3} \times \frac{3}{4} = \frac{6}{12} = \frac{1}{2}$ e.g., $7.2 \div 0.4$ as $7.2 \div 0.8 = 9$ so $7.2 \div 0.4 = 18$ (doubling and halving with place value).	**Fractions, Ratios, and Proportions by Re-unitising** The student chooses appropriately from a broad range of mental strategies to estimate answers and solve problems involving fractions, proportions, and ratios. These strategies involve using common factors, re-unitising of fractions, decimals and percentages, and finding relationships between and within ratios and simple rates. e.g., 6:9 as $\square$:24, $6 \times 1\frac{1}{2} = 9$, $\square \times 1\frac{1}{2} = 24$, $\square = 16$ (between unit multiplying); or $9 \times 2\frac{2}{3} = 24$, $6 \times 2\frac{2}{3} = 16$ (within unit multiplying) e.g., 65% of 24: 50% of 24 is 12, 10% of 24 is 2.4 so 5% is 1.2, $12 + 2.4 + 1.2 = 15.6$ (partitioning percentages).

The Number Framework – Knowledge

Stage	Number Identification	Number Sequence and Order	Grouping/Place Value	Basic Facts	Written Recording
Stage Zero: Emergent	The student identifies: • all of the numbers in the range 0–10.	The student says: • the number word sequences, forwards and backwards, in the range 0–10 at least; • the number before and after a given number in the range 0–10. The student orders: • numbers in the range 0–10.	The student instantly recognises: • patterns to 5, including finger patterns.		The student records: • numerals to match the sets they form.
Stages One, Two, Three: Counting from One	The student identifies: • all of the numbers in the range 0–20.	The student says: • the number word sequences, forwards and backwards, in the range 0–20; • the number before and after a given number in the range 0–20; • the skip-counting sequences, forwards and backwards, in the range 0–20 for twos and fives. The student orders: • numbers in the range 0–20.	The student knows: • groupings within 5, e.g., 2 and 3, 4 and 1; • groupings with 5, e.g., 5 and 1, 5 and 2, ...; • groupings within 10, e.g., 5 and 5, 4 and 6, ... etc; The student instantly recognises: • patterns to 10 (doubles and 5-based), including finger patterns.	The student recalls: • addition and subtraction facts to five, e.g., 2 + 1, 3 + 2, 4 – 2, ... etc; • doubles to 10, e.g., 3 + 3, 4 + 4, ... etc.	The student records: • the results of counting and operations using pictures and diagrams. The teacher and the student record: • the results of operations using symbols, e.g., the reading of five and two is seven is recorded as 5 and 2 is 7, 5 plus 2 equals 7, 5 + 2 = 7.

Stage	Number Identification	Number Sequence and Order	Grouping/Place Value	Basic Facts	Written Recording
Stage Four: Advanced Counting	The student identifies: • all of the numbers in the range 0–100; • symbols for halves, quarters, thirds, and fifths.	The student says: • the number word sequences, forwards and backwards, in the range 0–100; • the number before and after a given number in the range 0–100; • the skip-counting sequences, forwards and backwards, in the range 0–100 for twos, fives, and tens. The student orders: • numbers in the range 0–100.	The student knows: • groupings with 10, e.g., 10 and 2, 10 and 3, ... and the pattern of "-teens"; • groupings within 20, e.g., 12 and 8, 6 and 14; • the number of tens in decades, e.g., tens in 40, in 60.	The student recalls: • addition and subtraction facts to 10, e.g., 4 + 3, 6 + 2, 7 – 3, ...; • doubles to 20 and corresponding halves, e.g., 6 + 6, 7 + 7, $\frac{1}{2}$ of 14; • "ten and" facts, e.g., 10 + 4, 7 + 10 • multiples of 10 that add to 100, e.g., 30 + 70, 40 + 60.	The student records: • the results of mental addition and subtraction, using equations, e.g., 4 + 5 = 9, 8 – 3 = 5.
Stage Five: Early Additive	The student identifies: • all of the numbers in the range 0–1000; • symbols for the most common fractions, including at least halves, quarters, thirds, fifths, and tenths; • symbols for improper fractions, e.g., $\frac{5}{4}$.	The student says: • the number word sequences, forwards and backwards, by ones, tens, and hundreds in the range 0–1000; • the number 1, 10, 100 before and after a given number in the range 0–1000; • the skip-counting sequences, forwards and backwards, in the range 0–100 for twos, threes, fives, and tens. The student orders: • numbers in the range 0–1000; • fractions with like denominators, e.g., $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$, ... etc.	The student knows: • groupings within 100, e.g., 49 and 51 (particularly multiples of 5, e.g., 25 and 75); • groupings of two that are in numbers to 20, e.g., 8 groups of 2 in 17; • groupings of five in numbers to 50, e.g., 9 groups of 5 in 47; • groupings of ten that can be made from a three-digit number, e.g., tens in 763 is 76; • the number of hundreds in centuries and thousands, e.g., hundreds in 800 is 8 and in 4000 is 40. The student rounds: • three-digit whole numbers to the nearest 10 or 100 e.g., 561 rounded to the nearest 10 is 560 and to the nearest 100 is 600.	The student recalls: • addition facts to 20 and subtraction facts to 10, e.g., 7 + 5, 8 + 7, 9 – 6, ...; • multiplication facts for the 2, 5, and 10 times tables and the corresponding division facts; • multiples of 100 that add to 1000, e.g., 400 and 600, 300 and 700.	The student records: • the results of addition, subtraction, and multiplication calculations using equations, e.g., 35 + 24 = 59, 4 × 5 = 20, and diagrams, e.g., an empty number line.

Stage	Number Identification	Number Sequence and Order	Grouping/Place Value	Basic Facts	Written Recording
Stage Six: Advanced Additive–Early Multiplicative	The student identifies: • all of the numbers in the range 0–1 000 000; • decimals to three places; • symbols for any fraction including tenths, hundredths, thousandths, and improper fractions.	The student says: • the whole number word sequences, forwards and backwards, by ones, tens, hundreds, and thousands in the range 0–1 000 000; • the number 1, 10, 100, 1000 before and after a given whole number in the range 0–1 000 000; • forwards and backwards word sequences for halves, quarters, thirds, fifths, and tenths, e.g., $\frac{1}{3}$, $\frac{2}{3}$, 1, $\frac{4}{3}$, $\frac{5}{3}$, etc. • the decimal number word sequences, forwards and backwards, in tenths and hundredths. The student orders: • whole numbers in the range 0–1 000 000; • unit fractions for halves, thirds, quarters, fifths, and tenths.	The student knows: • groupings within 1000, e.g., 240 and 760, 498 and 502, ...; • groupings of two, three, five, and ten that are in numbers to 100 and finds the resulting remainders, e.g., threes in 17 is 5 with 2 remainder, fives in 48 is 9 with 3 remainder. • groupings of 10 and 100 that can be made from a four-digit number, e.g., tens in 4562 is 456 with 2 remainder, hundreds in 7894 is 78 with 94 remainder. • tenths and hundredths in decimals to two places, e.g. tenths in 7.2 is 72, hundredths in 2.84 is 284. The student rounds: • whole numbers to the nearest 10, 100, or 1000. • decimals with up to two decimal places to the nearest whole number, e.g., rounds 6.49 to 6, rounds 19.91 to 20.	The student recalls: • addition and subtraction facts up to 20, e.g., 9 + 5, 13 – 7; • multiplication basic facts up to the 10 times tables (10 × 10) and some corresponding division facts; • multiplication basic facts with tens, hundreds and thousands, e.g., 10 × 100 = 1000, 100 × 100 = 10 000	The student: • records the results of calculations using addition, subtraction, multiplication, and division equations, e.g., 349 + 452 = 350 + 451 = 801, e.g., 45 ÷ 9 = 5, • demonstrates the calculation on a number line or with a diagram. The student performs: • column addition and subtraction with whole numbers of up to four digits. e.g., $\begin{array}{r} {\scriptstyle 1\,1} \\ 476 \\ +\,285 \\ \hline 761 \end{array}$ $\begin{array}{r} {\scriptstyle 6\,9\,9\,1} \\ \cancel{700}0 \\ -\,586 \\ \hline 6414 \end{array}$

Stage	Number Identification	Number Sequence and Order	Grouping/Place Value	Basic Facts	Written Recording
◄······ Stage Seven: Advanced Multiplicative–Early Proportional ······►		The student says: • the decimal word sequences, forwards and backwards, by thousandths, hundredths, tenths, ones, tens, etc.; • the number one-thousandth, one-hundredth, one-tenth, one, ten, etc. before and after any given whole number. The student orders: • decimals to three places, e.g., 6.25 and 6.3; • fractions, including halves, thirds, quarters, fifths, tenths.	The student knows: • the groupings of numbers to 10 that are in numbers to 100 and finds the resulting remainders, e.g., sixes in 38, nines in 68; • the groupings of 10, 100, and 1000 that can be made from a number of up to seven digits, e.g., tens in 47 562, hundreds in 782 894, thousands in 2 785 671; • equivalent fractions for halves, thirds, quarters, fifths, and tenths with denominators up to 100 and up to 1000, e.g., 1 in 4 is equivalent to 25 in 100 or 250 in 1000. The student rounds: • whole numbers and decimals with up to two places to the nearest whole number or $\frac{1}{10}$, e.g., rounds 6.49 to 6.5 (nearest tenth).	The student recalls: • division basic facts up to the 10 times tables, e.g., 72 ÷ 8. • fraction <—> decimal <—> percentage conversions for halves, thirds, quarters, fifths, and tenths, e.g., $\frac{3}{4} = 0.75 = 75\%$. The student knows: • divisibility rules for 2, 3, 5, 9, and 10, e.g., 471 is divisible by 3 since 4 + 7 + 1 = 12; • square numbers to 100 and the corresponding roots, e.g., $7^2 = 49$, $\sqrt{49} = 7$. The student identifies: • factors of numbers to 100, including prime numbers, e.g., factors of 36 = {1, 2, 3, 4, 6, 9, 12, 18, 36}; • common multiples of numbers to 10, e.g., 35, 70, 105, ... are common multiples of 5 and 7.	The student records: • the results of calculations using equations, e.g., 6 × 28 = 168, and diagrams, e.g., empty number line. The student performs: • column addition and subtraction for whole numbers; • short multiplication and division of a three-digit whole number by a single-digit number. e.g., $\begin{array}{r} {}^{5\,2}\\ 473 \\ \times\ 8 \\ \hline 3784 \end{array}$ $\begin{array}{r} 784 \\ 7\overline{)\overset{5\,2}{5488}} \end{array}$

Stage	Number Identification	Number Sequence and Order	Grouping/Place Value	Basic Facts	Written Recording
Stage Eight: Advanced Proportional		The student says: • the decimal word sequences, forwards and backwards, by thousandths, hundredths, tenths, ones, tens, etc., starting at any decimal number; • the number one-thousandth, one-hundredth, one-tenth, one, ten, etc. before and after any given decimal number. The student orders: • fractions, decimals, and percentages.	The student knows: • the number of tenths, hundredths, and one-thousandths that are in numbers of up to three decimal places, e.g., tenths in 45.6 is 456, hundredths in 9.03 is 903, thousandths in 8.502 is 8502; • what happens when a whole number or decimal is multiplied or divided by a power of 10, e.g., 4.5×100, $67.3 \div 10$. The student rounds: • decimals to the nearest 100, 10, 1, $\frac{1}{10}$, or $\frac{1}{100}$, e.g., rounding 5234 to the nearest 100 gives 5200.	The student recalls: • fraction <—> decimal <—> percentage conversions for given fractions and decimals, e.g., $\frac{9}{8} = 1.125 = 112.5\%$. The student knows: • divisibility rules for 2, 3, 4, 5, 6, 8, and 10, e.g., 5 632 is divisible by 8 since 632 is divisible by 8, e.g., 756 is divisible by 3 and 9 as its digital root is 9; • simple powers of numbers to 10, e.g., $2^4 = 16$, $5^3 = 125$ The student identifies: • common factors of numbers to 100, including the highest common factor, e.g., common factors of 48 and 64 = {1, 2, 4, 8, 16}; • least common multiples of numbers to 10, e.g., 24 is the least common multiple of 6 and 8.	The student records: • the results of calculations using equations, e.g., $\frac{3}{4} \times 28 = 21$, and diagrams, e.g., double number line. The student performs: • column addition and subtraction for whole numbers, and decimals to three places; • short multiplication and division of whole numbers and decimals by single-digit numbers, e.g., $\begin{array}{r} 14.65 \\ 4\overline{)5\overset{1}{8}.\overset{2}{6}\overset{2}{0}} \end{array}$ • multiplication of three- or four-digit whole numbers by two-digit whole numbers, e.g., $\begin{array}{r} {\scriptstyle 21} \\ {\scriptstyle 52} \\ 763 \\ \times\ 49 \\ \hline 6867 \\ \underline{30520} \\ 37387 \end{array}$

ACKNOWLEDGMENTS

The Ministry of Education wishes to acknowledge the following people and organisations for their contribution towards the development of this handbook.

THE PARTICIPANTS:

The New Zealand numeracy project personnel – facilitators and principals, teachers, and children from more than two thousand New Zealand schools who contributed to this handbook through their participation in the numeracy development projects from 2000–2006.

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REVISED EDITION 2007:

People who contributed to the 2007 revision of pages 4–13 include Associate Professor Jenny Young-Loveridge (The University of Waikato), Malcolm Hyland (Ministry of Education), Gaynor Terrill and Vince Wright (The University of Waikato School Support Services), other numeracy facilitators throughout New Zealand, and Susan Roche (Learning Media Ltd).

In addition, the Ministry of Education wishes to acknowledge Professor Bob Wright (Southem Cross University, Lismore, NSW), Dr Noel Thomas (Charles Sturt University, Bathhurst, NSW), Dr Koeno Gravemeijer (Freudenthal Institute, Utrecht, Netherlands), Jim Martland (The University of Manchester, UK), and Susan Lamon (Marquette University, USA).

The Ministry of Education also wishes to acknowledge The New South Wales Department of Education and Training for permission to trial *Count Me In Too* in 2000 through a one-year arrangement. The findings from the use of this pilot project informed the development of the numeracy policy in New Zealand.

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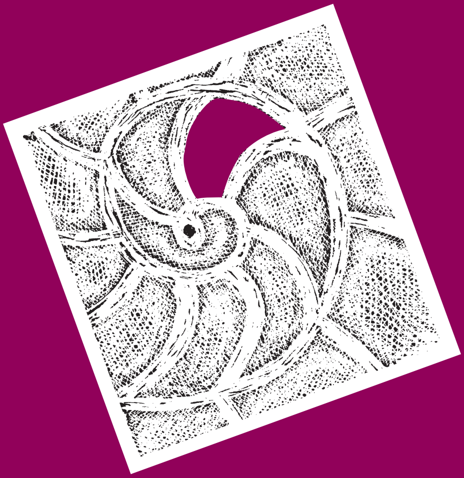
Book 1



To be numerate is to have the ability and inclination to use mathematics effectively — at home, at work and in the community.

ISBN 978-0-7903-1668-0

New Zealand Government