AC
EA
AA
AM
AP

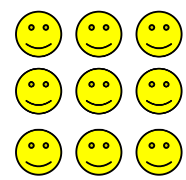
Thinking Inside the Square

We are developing thinking skills of connecting and analysing.
We are exploring some properties of square numbers.
We are developing number strategies using square numbers.
We are practising the communication of explanation.

Exercise 1 – A Square Number

Work in pairs, without calculators.
This exercise is to make sure you know all the basic square numbers.

The Idea
A number multiplied by itself makes a square number.



We should understand

3 x 3 = 9 where 9 is the square number and what it looks like.
The number 9 is the square of the number 3.
We can also write $9 = 3^2$ and read this as "9 equals 3 squared".
We can also say "3 squared equals 9" because that is how the "=" sign works.

Multiply these

1) 2 x 2 (2) 5 x 5 (3) 1 x 1

4) 4 x 4 (5) 9 x 9 (6) 10 x 10

7) 7 x 7 (8) 6 x 6 (9) 8 x 8

Square these numbers

10) 7 (11) 8 (12) 9

13) 6 (14) 5 (15) 2

16) 4 (17) 0 (18) 1

Task of the century
On a 100 board put counters on all the square numbers. Do you notice any patterns?

Write the squares of these numbers underneath.

1	2	3	4	5	6	7	8	9

Write here any odd thing you have you noticed. Explain your observation.

Exercise 2 – An Associative Problem

Work in pairs, without calculators.

This exercise will help you know some the squares of the decades to 100 and uses your knowledge of the basic squares.

Place Value Revision

You should know that 20 = 2 x 10. That is what two-tens or twenty means.

Expand these numbers in the same way. Eg 80 = 8 x 10

1)	30	(2)	60	(3)	70
4)	50	(5)	90	(6)	10
7)	40	(8)	20	(9)	80

You may not know how to multiply 20 x 20 and get 400, but here is one way.

20 x 20 --> 2 x 10 x 2 x 10 --> 2 x 2 x 10 x 10 --> 4 x 100 --> 400

We collect the 2's and the 10's together. Why?

400 is the square of 20

20 squared is 400

$20^2 = 400$

Square these numbers.

10)	30	(11)	60	(12)	70
13)	50	(14)	90	(15)	10
16)	40	(17)	20	(18)	80

Use the same expanded numeral idea to square these numbers

19)	300	(20)	600	(21)	700
22)	500	(23)	900	(24)	100
25)	400	(26)	200	(27)	800

And lastly these really big numbers.

28)	3000	(29)	60,000	(30)	700,000
31)	10,000,000	(32)	50,000,000	(33)	700,000,000
34)	a billion	(35)	a quadrillion	(36)	a googol

What happens to the number of zeros (place-holders) when the number is squared?

Exercise 3 – A Distribution Problem

Work in pairs, without calculators.
This exercise will help you know and remember some of the squares bigger than 10.

$25 \times 25 = (20 + 5) \times (20 + 5)$ but what does this equal?

Why did we choose (20 + 5) and not something else like (17 + 8) which also equals 25?

The array model of multiplication shows what happens and how to get the answer.

X	20	5
20	20x20=400	100
5	100	25

The answer to 25 x 25 is equal to <u>400</u> + <u>2 lots of 100</u> + <u>25</u> = 625.

Explain where each of the numbers comes from to a friend.

Now use this model to square all of these numbers.

1)	15 = (10 + 5)	(2)	12 = (10 + 2)	(3)	17 = (10 + 7)
4)	11	(5)	14	(6)	19
7)	18	(8)	16	(9)	15
10)	13	(11)	21	(12)	45

What do you notice in these answers?

Write the squares of these numbers underneath.

11	12	13	14	15	16	17	18	19

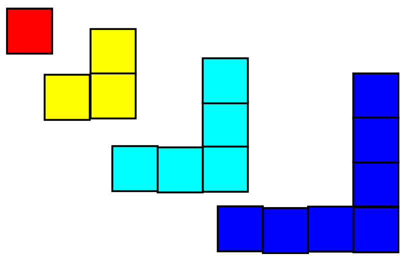
Invent a game that will help you to memorise these square numbers.

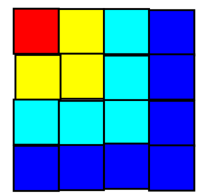
Exercise 4 – A Square is an Odd Combination

Work in pairs.
You need multilink blocks.

The odd numbers are quite closely related to the square numbers. This exercise explores this interesting relationship. It is a very useful to have this understanding.

Firstly make a 1 square then add an L shape of 3 in a different colour. Then add L5 and so on always using a different colour.





What do you notice?

Apart from the lovely colours and having fun with the blocks; it is interesting to notice that the odd numbers add up to make the square numbers.

$$1 + 3 = 2^2$$

$$1 + 3 + 5 = 3^2$$

$$1 + 3 + 5 + 7 = 4^2$$

What do you notice in this pattern?

What do you think the first 5 odd numbers add up to? Explain…and check.

What would the first 10 odd numbers add up to? Explain…and check.

Can you generalise this idea and say what any number of odd numbers would sum to?

Exercise 5 – Odd Patterns.

Work in pairs.

The set of Odd Numbers = { 1, 3 ,5, 7, 9, 11, 13, 15, 17, 19, 21, 23, …}

There is not enough paper in the universe to write this set out completely. In fact there is not enough ink either. Nor is the universe large enough to stuff all the paper you would need. The odd number set is an example of an **infinite** set.

Fill the gaps and extend this pattern.

Summing the Odds	**Sum**	**A Pattern**
1	1	1^2
1 + 3	4	2^2
1 + 3 + 5		3^2
1 + 3 + 5 + 7	16	
	36	
What do you notice?		

Another way to look at the odd numbers. Fill in the gaps and extend the pattern

Curious Odd Numbers	**Sum**	**A Cube Pattern**
1	1	$1 \times 1 \times 1 = 1^3$
3 + 5	8	
	27	$3 \times 3 \times 3 = 3^3$
13 + 15 + 17 + 19		$4 \times 4 \times 4 = 4^3$

In this pattern what is the middle number in the 3rd row?

What is the middle number in the 11th row?

What do the numbers in the 5th row add up to?

What do the numbers in the 11th row add up to?

Now explore and see if you can discover an odd pattern.

Exercise 6 – Square Stepping Stones

Work in pairs, without calculators.

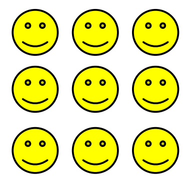
Here is a problem to solve.

If you know that 12 x 12 = 144,
how can you figure out what the answer to 13 x 13 is?

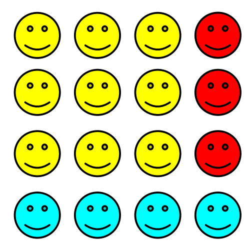
To solve this we will use the "make it smaller" strategy.

If you know that 3 x 3 = 9, how can you know what 4x4 is?

Here is 3 x 3. To make 4 x 4 from the 3x3 I copied a column of 3 and pasted them as "red" smiley faces. Then I copied a row of 4 and pasted them as "aqua" smiley faces.

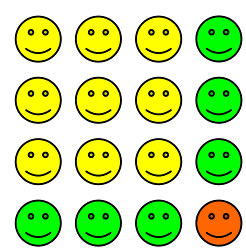


Looking from left to right;
I added a red face to each row and then a row of 4.



4 x 4 = 3 x 3 + 3 x 1 + 1 x 4
= 3 lots of 3 and 3 lots of 1 and 1 lot of 4
= 3 lots of 4 and 1 lot of 4
= 4 lots of 4
= 4 x 4

Another way of getting to the answer is add two more lots of 3 and then 1 for the missing corner.



Can you see any other ways to make the 4 x 4?

Use one of these methods to find all the squares to 23 starting with the 9 x 9 = 81.

1	9 x 9	81	6	14 x 14		11	19 x 19	
2	10 x 10		7	15 x 15		12	20 x 20	
3	11 x 11		8	16 x 16		13	21 x 21	
4	12 x 12		9	17 x 17		14	22 x 22	
5	13 x 13		10	18 x 18		15	23 x 23	

If 32 x 32 = 1024, what is 33 x 33? ______________________________

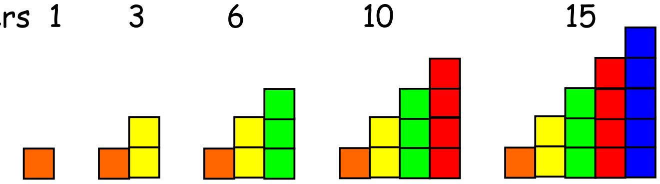
If 222 x 222 = 49284, what is 223 x 223? ______________________________

Exercise 7 – A Square is Triangular!

Work in pairs.
You need multilink blocks.

This exercise explores the way triangular numbers combine and make the squares.

Firstly triangular numbers 1 3 6 10 15 … … …



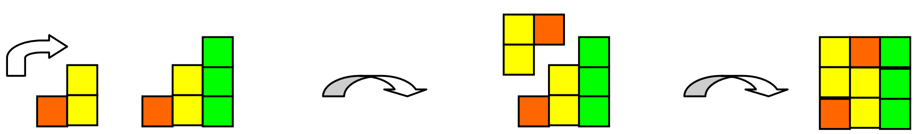
Why do you think they are called triangular numbers?

Above are models for the first 5 triangular numbers.
We will call them T1, T2, T3, T4 and T5. Note that T3 = 1 + 2 + 3 = 6 blocks.

What does T8 = _________________________ ?

What does T10 = _________________________ ?

Now for an interesting twist.
If we take T2 and turn it over, it fits into T3 and makes a 3 x 3 square.



Make models using the multilink blocks and explore what happens when two consecutive triangular numbers are combined in this way.

Record here what you found out.

Check using the numbers above that two consecutive triangular numbers make a square number and write the results here.

What two triangular numbers make up the square number 12 x 12? ______ ________
What two triangular numbers make up the square number 289 ? ________ ________
What two triangular numbers make up the square number 10,000 ? _______ _______

Exercise 8 – Connecting Squares, Odds and Triangular

Work in pairs.
You will need multilink blocks.

This exercise shows how ideas in mathematics are connected.

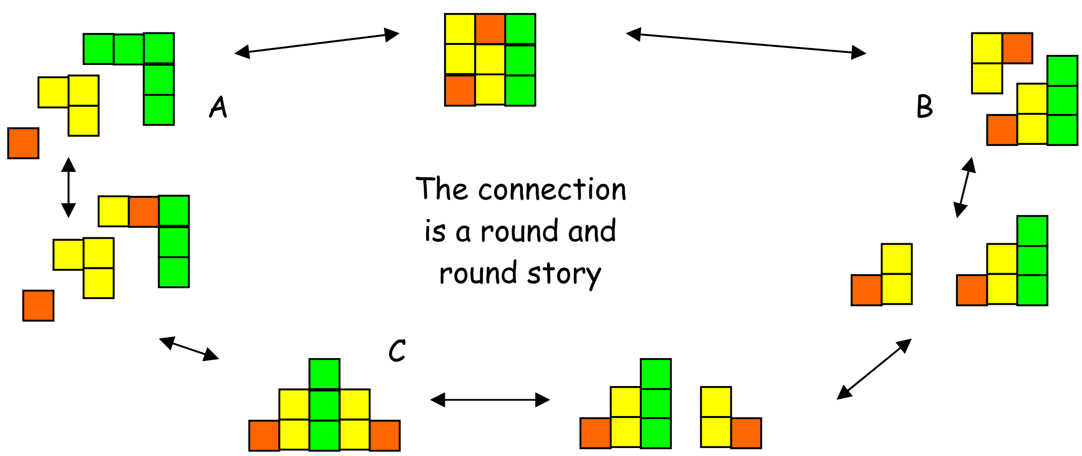
The Connection

A square is the sum of some odd numbers. Eg $3^2 = 9 = 1 + 3 + 5$

Two triangular numbers join to make a square number. Eg $T2 + T3 = 3 + 6 = 9 = 3^2$

So the odd numbers **must** be connected to the triangular numbers.

How are they connected?

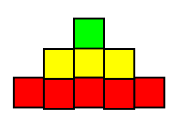


Study the diagrams above. Look for links from one to the next. Go around clockwise and then the other way.

Now do it! Make a set of these with the multilink blocks and transform them from one shape to the next.
At A you can see how 1 + 3 + 5 join together to form the 3 x 3 square.
At B you can see how the square is made up from the T2 and T3 triangular numbers.
Look carefully at C and find the link between T2 and T3 and the odd numbers.

Write here what you see.

The **mind shift** you need to do is to look at the shape in a different way. Look across the layers and see the 1 in the top layer, the 3 in the next layer and then the 5 in the bottom layer. This is an important problem solving technique to remember and use.



Last thought. The odd numbers are closely related to the even numbers! Hmmmm.

Exercise 9 – Squaring Down

Work in pairs, without calculators.

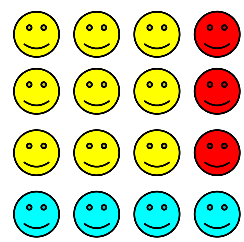
Here is another problem to solve.

If you know that 13 x 13 = 169,
how can you figure out what the answer to 12 x 12 is?

To solve this we will use the "make it smaller" problem solving strategy.

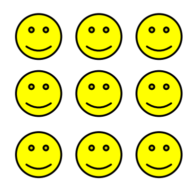
If you know that 4x4 = 16, how can you know what 3x3 is?

Here is 4 x 4. To make 3 x 3 from the 4 x4 I take away the "aqua" row of 4 and then the "red" column of 3.

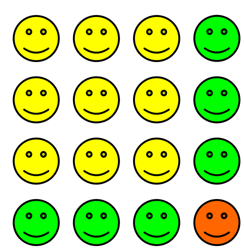


I subtracted a row of 4 and then a column of 3.

4 x 4	= 3 x 3 + 3 + 4
4 x 4 - 4	= 3 lots of 3 and 3
4 x 4 – 4 - 3	= 3 lots of 3
	= 3 x 3



Another way of getting to the answer is subtract two lots of 3 and then subtract the lonely 1.



Why do we always subtract an odd and an even number?

Use any method to find these squares.

1) Use 30 x 30 = 900 to solve 29 x29 =
2) Use 25 x 25 = 625 to solve 24 x 24 =
3) Use 70 x 70 = 4900 to solve 69 x 69 =
4) Use 75 x 75 = 5625 to solve 74 x 74 =
5) Use 100 x 100 = 10000 to solve 99 x 99 =
6) Use 250 x 250 = 62500 to solve 249 x 249 =
7) Use 50 x 50 = 2500 to solve 49 x 49 =
8) Use 1000 x 1000 = 1000000 to solve 999 x 999 =
9) Use 37 x 37 = 1396 to solve 36 x 36 =

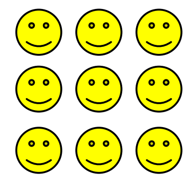
Exercise 10 – Squares that Differ by Two

Work in pairs, without calculators.

Here is a bigger problem to solve.

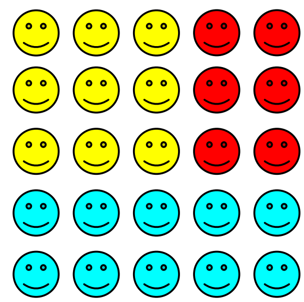
If you know that 12 x 12 = 144,
how can you figure out what the answer to 14 x 14 is?

To solve this we will us the "make it smaller" strategy again.

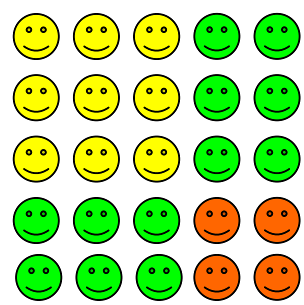


If you know that 3 x 3 = 9, how can you know what 5 x 5 is?

Here is 3 x 3. To make 5 x 5 from the 3 x 3 I copied two columns of 3 and pasted them as "red" smiley faces. Then I copied two rows of 5 and pasted them as "aqua.
Looking at this across the faces or left to right.
I added two "red" smileys to each of the 3 rows and then added two rows of 5.



5 x 5 = 3 x 3 + 3 x 2 + 2 x 5
= 3 lots of 3 and 3 lots of 2 and 2 lots of 5
= 3 lots of 5 and 2 lots of 5
= 5 lots of 5



Another way of getting to the answer is add two more green faces for each row and for each column and then 2 x 2 for the missing corner.

There are other ways to look at this problem.

Use one of these methods to increase all these squares.

1) Use 9 x 9 = 81 to solve 11 x 11 =
2) Use 22 x 22 = 484 to solve 24 x 24 =
3) Use 42 x 42 = 1764 to solve 44 x 44 =
4) Use 51 x 51 = 2809 to solve 53 x 53 =
5) Use 89 x 89 = 7921 to solve 91 x 91 =
6) Use 120 x 120 = 14400 to solve 122 x 122 =
7) Use 12 x 12 = 144 to solve 14 x 14 =
8) Use 222 x 22 = 49284 to solve 224 x 224 =
9) What could you do if the increase was more? Eg 20 x 20 = 400 so what is 27 x 27?

Exercise 11 – A Curious Square Trick

Work in pairs, without calculators.

65 x 65 = 4225

We can calculate the answer to similar numbers like 35, 45, 95 very easily.
It always ends in 25
The hundreds digits are 6 x (6+1) = 42
Joining these gives 4225

Work out these using the rule in the box.

1) 25 x 25
2) 35 x 35
3) 55 x 55
4) 75 x 75
5) 15 x 15
6) 85 x 85

It even works on bigger numbers

165 x 165

Here the hundreds is
16 x 17 = 16 x 16 + 16= 282
Answer is 28225
But we do need to know how to multiple consecutive numbers like 16x17

Work out these big squares

7) 105 x 105
8) 115 x 115
9) 125 x 125
10) 155 x 155
11) 185 x 185
12) 195 x 195

But wait…why does it work?

So far we are only doing numbers.
We are not doing mathematics until we know why it works;
And can explain it to someone.

Investigate this problem and write your thoughtful answer to explain why it works.

Exercise 12 – Introducing the Square Root

Work in pairs.
You need a calculator.

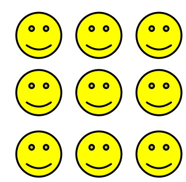
This exercise is an introduction to the “inverse” of squaring a number.

The Basic Idea
12 x 12 = 144
Here the 144 is the square of 12
But we can look at this another way.
Here the 12 is the square root of 144.
Why it is called a square root is another puzzle!

A square root is the number which when you multiply it by itself will give you the original number!

An example is 9 which has a square root of 3, because 3 x 3 = 9.

Arranging the number 9 in a square pattern illustrates where the 3 comes from. It is the length of one side of the square.



How about the number 10?
Can you think of a number that when you square it will give you 10?
How about the number 16?
Can you think of a number that when you square it will give you 16?

$\boxed{\sqrt{}}$

Sometimes it is very easy to get the square root but usually it is very difficult. That is one good reason why there is a “square root” button on every calculator.

Estimate the square root answers to these problems. Some easy, some hard.

1) 25	2) 24	3) 26
4) 64	5) 60	6) 70

Use your calculator to find the exact answers.

Now for a bewildering insight. See if you can explain this!
Estimate the square root answers to these problems. Some easy, some hard.

7) 250	8) 240	9) 260
10) 640	11) 600	12) 700

Use your calculator to find the exact answers.

Why were these answers not related to the other 6 problems?

Exercise 13 – Revision of “n”

You will need a long paper and a pencil.

Draw an empty number line like this

Now mark the number 0 or zero near the middle.
Now place marks for all of these numbers.

(a) n
(b) the number one less than n
(c) the number 3 less than n
(d) the number 5 less than n
(e) the number 4 more than n
(f) the number 1 bigger than n
(g) the number 2 more than n
(h) the number halfway between 0 and n +1
(i) the number 2
(j) the number 2n
(k) the number 2n +2
(l) the number 2n + 2n
(m) the number that is $\frac{1}{4}$of n smaller than n
(n) –n
(o) -1
(p) 1 - n
(q) –n - 1
(r) –n + 1
(s) -n + 2
(t) -2n
(u) n^2
(v) n^3
(w) square root of n
(x) 1 nth
(y) n nths
(z) the sum of n numbers

Thinking Inside the Square
Answers

Exercise 1

1)	4	(2)	25	(3)	1
4)	16	(5)	81	(6)	100
7)	49	(8)	36	(9)	64
10)	49	(11)	64	(12)	81
13)	36	(14)	25	(15)	4
16)	16	(17)	0	(18)	1

Squares of the numbers 1 to 9 are 1, 4, 9, 16, 25, 36, 49, 64, 81
Noticed? The difference between the squares are odd. The squares are odd, even, odd, even…
The differences are odd because always we add an odd and an even number which is odd and always two more. The odd/even pattern is the result of squaring an odd or even number which arose from add + add = even , even + odd = odd and so on. The oddness toggles. Other patterns possible.

Exercise 2

1)	3x10	(2)	6x10	(3)	7x10
4)	5x10	(5)	9x10	(6)	1x10
7)	4x10	(8)	2x10	(9)	8x10
10)	900	(11)	3600	(12)	4900
13)	2500	(14)	8100	(15)	100
16)	1600	(17)	400	(18)	6400
19)	90000	(20)	360000	(21)	490000
22)	250000	(23)	810000	(24)	10000
25)	160000	(26)	40000	(27)	640000
28)	9000000 (6 zeros)	(29)	3600000000 (8 zeros)		
(30)	490000000000 (10 zeros)	31)	100000000000000 (14 zeros)		
(32)	2500000000000000 (14 zeros)	(33)	490000000000000000 (16 zeros)		
34)	1000000000000000000 (18 zeros)				
(35)	1000000000000000000000000000000000000 (30 zeros)				

(36) A googol is 1 followed by 100 zeros so a googol squared is :
10000000000000000000000000000000000000000000000000000000000000000000000000000000
00000000000000000000000000000000000000000000000000000000000000000000000000000000
00000000000000000000000000000000000000000 1 followed by 200 zeros!
A googol is 1 followed by 100 zeros.
The number of zeros or placeholders doubles when a number is squared.

Exercise 3

Choosing 20 + 5 is better than 17+8 because it is easier to deal with.

1)	100 + 50 + 50 + 25	(2)	100 + 20 + 20 + 4	(3)	100 + 70 + 70 + 49
4)	100 + 10 + 10 + 1	(5)	100 + 40 + 40 + 16	(6)	100 + 80 + 80 + 81
7)	100 + 80 + 80 + 64	(8)	100 + 30 + 30 + 36	(9)	100 + 50 + 50 + 25
10)	100 + 30 + 30 + 9	(11)	400 + 20 + 20 + 1		
(12)	1600 + 200 + 200 + 25 or another way.				

The answers always have 4 parts.
Squares of numbers 11 to 19 are 121, 144, 169, 196, 225, 256, 289, 324, 361
Games can include pickup memory, snakes mystery cards, square dominos, snap.

Exercise 4

Students should notice the L shape gets bigger by two each time and explain why.
Firt 4 odd numbers must add to 16.
First 10 numbers will add to 100.
The sum from 1 of any n odd numbers is nxn or n squared.

Exercise 5

The infinite set may need explaining. The number of grains of sand on all the beaches in the world is not an infinite set but it is uncountable. The number of odd numbers is infinite because there is always a bigger one than the biggest one you can name.

Pattern 1
This shows the sum of odd numbers, the sum, and the sum written as a square.
Last row = 1 = 3 + 5 + 7 + 9 + 11 + 13 or 7 terms, 49, 7^2

Pattern 2
This is as follows

1	1	1^3
3+5	8	2^3
7+9+11	27	3^3
13+15+17+19	64	4^3
21+23+25+27+29+31	125	5^3
33+35+37+39+41+43+45	216	6^3
47+49+51+53+55+57+59+61	343	7^3
63+65+67+69+71+73+75+77+79	512	8^3

middle in 3rd row is 9 or 3 squared
middle in 11th row is 11 squared or 121
5th row add to 5x5x5 = 125
1th row adds to 11x11x11 = 1331 (nice link here to Pascals triangle)
Other patterns exist! Look at the digit place and how it changes downwards. Odd/even in cubes!

Exercise 6

Other ways are possible to make 4x4. 2 groups of 2x4 or 4 groups of 2x2
Using method 1

1)	81	(2)	81 + 9 + 10 = 100	(3)	100+10+11=121
4)	121+11+12=144	(5)	144+12+13= 169	(6)	169+13+14=196
7)	195+14+15=225	(8)	225+15+16=256	(9)	256+16+17=289

Using method 2

10)	289+2x17 =1=324	(11)	324+2x18=1=361	(12)	361+2x19=1=400
13)	400+2x20+1=441	(14)	441+2x21+1=484	(15)	484+2x22+1=529

33x33=1024+32+33=1089, using method 1
223x223=49284+2x222+1=49729, using method 2

Exercise 7

The first 12 triangular numbers are 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78…
T8 = 36 and T10 =55
Recording will be various but should say it always makes a square.
T11+T12 = 12 x 12
289 = 17x17 so T16+T17 will make 289

10000 = 100x100 so T99 + T100 will make 10,000.
A curious extension is “Are there any triangular numbers that are also square?” to which the answer is of course “no”. A triangular number can be rearranged to make a rectangle but never a square and the closest rectangle will always have a side that is just one bigger than the other.

Exercise 8

Study, various.
What do you see is various.
The main purpose here is to build the models and feel how they transform from one to the other. Remaking the T2 with the colours shifted allows the odd numbers to be more clearly seen. The bottom diagram shows this.

Exercise 9

If we subtract an odd then we must subtract an even and if we subtract an even then we must subtract an odd because the shape is always reduced by 1.

1) 900-30-29=841
2) 625-25-24=576
3) 4900-70-69=4761
4) 5625-75-74 = 5476
5) 10000-100-99=9801
6) 62500-250-249=62001
7) 49x49=2500-50-49 = 2401
8) 999x999=1000000-1000-999=998001
9) 36x36=1369-37-36=1296

Exercise 10

1) 81+2x9x2+2x2=121
2) 484+2x22x2+2x2=576
3) 1764+2x42x2+2x2=1936
4) 2601 +2x51x2 + 2x2 =2809
5) 7921+2x89x2+2x2= 8281
6) 14400+2x120x2+2x2=14884
7) 34x34=1024 + 2x32x2+4 = 1156
8) 224x224 = 49284 + double double 222 + 4 = 50176
9) If the increase is n then the next square is 2 x n x the original number plus n x n. Since 20 x 20 = 400, so 27x27 = 400 + 2x7x20+7x7= 729

	81	(2)
3)	484	4)
5)	1764	(6)
7)	2601	(8)
9)	7921	10)
11)	14400	(12)

Exercise 11

1)	2x300+25=625	(2)	3x400+25=1225	(3)	5x600+25=3025
4)	7x800+25=5625	(5)	1x200+25=1225	(6)	8x900+25=7225
7)	10x1100+25=11025	(8)	11x1200+25=13225	(9)	12x1300+25=15625
10)	15x1600+25=24025	(11)	18x1900+25=34225	(12)	19x2000+25= 38025

Why does it work?
Using the array model we see the the calculation is;

$100xn^2 + 2x50xn +25= 100(n^2 + n) +25 = = 100x(n(n+1) +25$
where the n(n+1) is the product of two consecutive numbers
and the 100x pushes this into the hundreds column
which leaves the tens and units column clear for the 25.
A neat trick…can you find another?

Exercise 12

1)	5	(2)	a bit less than 5	(3)	a bit more than 5
4)	8	(5)	a bit less than 8	(6)	a bit more than 8
7)	nothing like 50	(8)	nothing like 50	(9)	nothing like 50
10)	nothing like 80	(11)	nothing like 80	(12)	nothing like 80

Exercise 13

Various answers. As long as the student can explain why the n is where it is placed and the explanation fits then the answer is OK.
Eg n can be to the left of 0 because n can be any where. 2n must be twice as far to the left if that is the case.
The notion of what the expression means is the important part. In this example "twice as far".