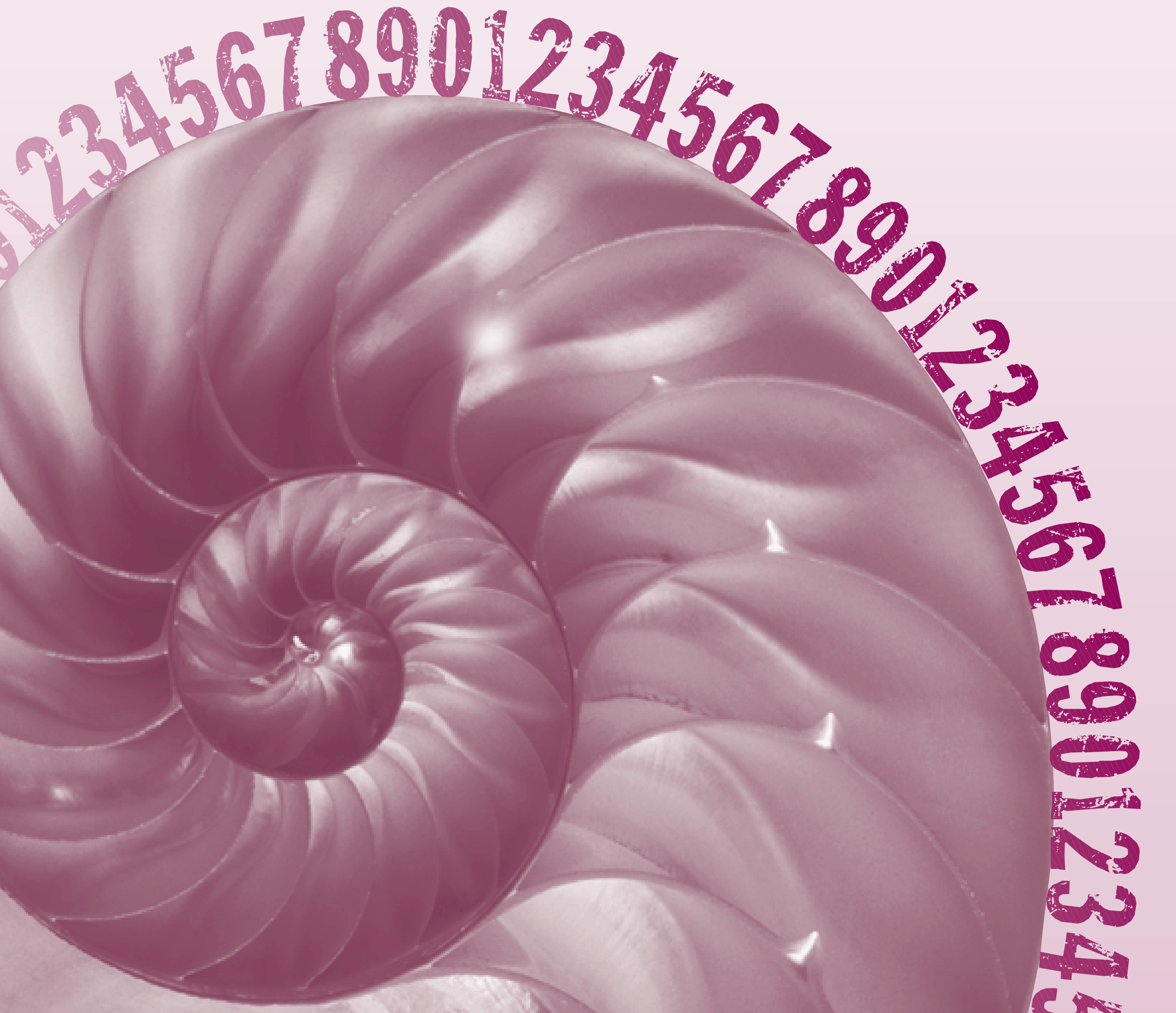
MINISTRY OF EDUCATION

Te Tāhuhu o te Mātauranga

Book 9
Teaching Number through Measurement, Geometry, Algebra and Statistics



Numeracy Professional Development Projects (Draft)

THE NUMBER FRAMEWORK AND MEASUREMENT, GEOMETRY, ALGEBRA AND STATISTICS

The Numeracy Professional Development Projects place a strong emphasis on students gaining an understanding of the number system. This is aligned with trends in modern mathematics education internationally. At all levels of schooling, teachers should encourage students to explore, describe and generalise structures and relationships through a range of mathematical activities.

Research about quality teaching[1] shows that students learn most quickly when they have opportunities to identify and resolve discrepancies between their current understandings and new information. The careful selection of related problems or investigations and the creation of a comfortable classroom climate in which all students can share their mathematical ideas are fundamental to improving achievement. The promotion of creative and efficient recording strategies can also greatly assist students in developing, generalising and communicating their ideas.

The Ministry of Education has adopted the following definition of numeracy: "to be numerate is to have the ability and inclination to use mathematics effectively in our lives – at home, at work, and in the community."[2] Numerate students are able to apply their number understanding to a range of contexts, from the other strands in mathematics, from other essential learning areas and from situations in their daily life, both real and imaginary.

This book makes explicit links between students' number knowledge and strategies and their ability to solve problems in measurement, geometry, algebra and statistics. These strands require students to make connections between their spatial visualisation and their ability to quantify (reason numerically). While strong number sense is not sufficient in itself for students to solve problems effectively in measurement, geometry and statistics, it is fundamental to success.

For example, students' ability to use continuous scales in measurement is critically dependent on their understanding of the number system, particularly of decimals. Their capacity to generalise how tessellations work involves understanding the angle concept. This requires both spatial and numeric reasoning. Statistical inquiry is becoming increasingly oriented towards finding relationships within existing data sets. Computer technology is providing powerful tools that allow students to explore these data sets, using a variety of representations. Critical use of these representations also requires both spatial and numeric reasoning.

This book aims to provide teachers with developmental links between the Number Framework and progressions in the different strands of *Mathematics in the New Zealand Curriculum*. In measurement and algebra, these links are very clear. In statistics and geometry, they are less defined. The lesson examples demonstrate how teachers can develop students' ability to generalise mathematically using contexts from the strands of *Mathematics in the New Zealand Curriculum.*

1 Alton-Lee, A. (2003). *Quality Teaching for Diverse Students in Schooling: Best Evidence Synthesis.* Wellington: Ministry of Education.

2 Quoted in *Curriculum Update* 45 (February 2001), page 1.

Numeracy Professional Development Projects 2007 (Draft)
Published by the Ministry of Education.
PO Box 1666, Wellington, New Zealand.

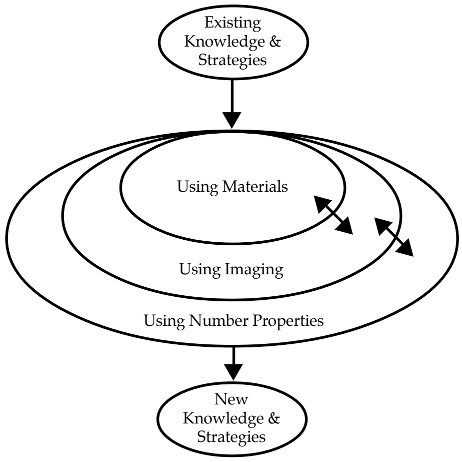
ISBN 0 478 13212 3
Dewey number 372.7
Topic Dewey number 510
Item number 13212

This book is also available on the New Zealand Maths website, at www.nzmaths.co.nz/Numeracy/2007numPDFs/pdfs.aspx

Linking the Number Framework with the Strands of the Mathematics Curriculum

This book is designed to provide links between students' development in the Number Framework and their capacity to solve problems in the different strands of *Mathematics in the New Zealand Curriculum.*

The teaching model shown below, which is taken from *Book 3: Getting Started,* emphasises imaging as an essential link between the students' manipulation of materials and their generalisation of number properties. There is a growing consensus that students' ability to hold and manipulate high quality images of objects is the most important factor in spatial visualisation.



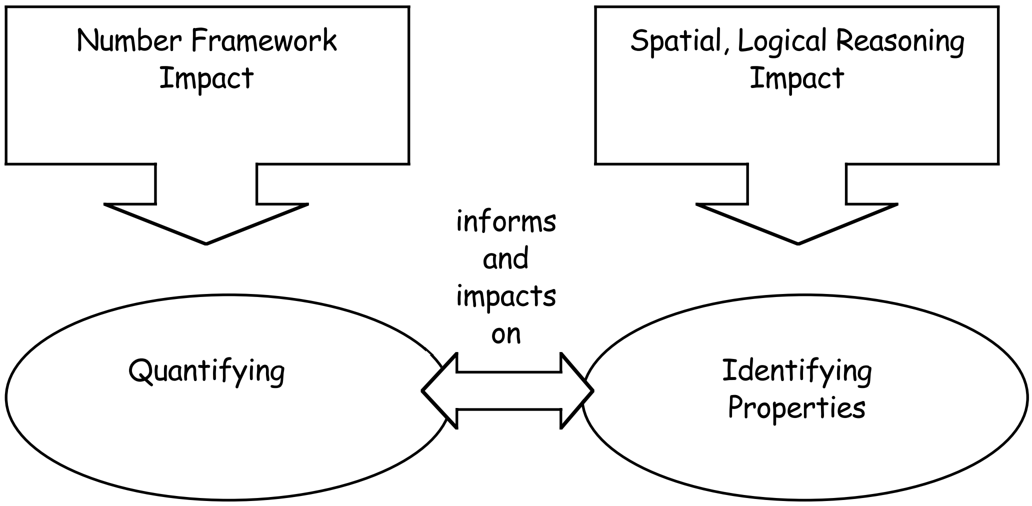
It is reasonable to expect that implementing this teaching model will assist students to develop their spatial visualisation and that teaching with an emphasis on spatial visualisation will greatly assist students to image actions on materials.

Spatial visualisation has a key role in reasoning within the different strands of the mathematics curriculum. In the measurement of volume, for example, students need to recognise that cubes can fill up a space and that some of the cubes filling a box may be hidden from their view.

In algebra, a geometric pattern may provide a sequence of numbers or a function and also give strong clues as to how the relationships can be generalised.

In statistics, there is an increasing emphasis on interpreting graphic displays, especially those generated by computers, rather than on processing data in a numeric form. Attending to the spatial and measurement features of a display, for example, scale, points and lines, is critical to successful interpretation.

The diagram below depicts the relationship between students' number knowledge and strategy and their proficiency in the different strands of the mathematics curriculum.



This diagram suggests that there is a dynamic way in which spatial visualisation and quantification inform and impact on each other. Students' number strategy impacts directly on their ability to quantify measurement units. For example, a student at the One-to-one Counting stage is likely to count the number of squares in an array one at a time, whereas a student at the Advanced Counting stage may use skip-counting. A student at the Advanced Additive stage may use multiplication to quantify the number of squares.

The exercise of quantifying units has an impact on how students perceive space. For example, much of the geometry of shapes and solids and of direction and movement depends on students' understanding the nature of angles. To teach the angle concept successfully, it is crucial to first support and enhance the act of quantifying (measuring) angles in degrees by spatial exploration and debating what constitutes an angle.

The Ministry of Education has adopted the following definition of numeracy: "to be numerate is to have the ability and inclination to use mathematics effectively in our lives – at home, at work, and in the community."[3] The definition of numeracy also states that the development of strong number sense is the key learning goal in mathematics for the early primary years but that, in the middle and upper stages of schooling, number increasingly becomes a tool to be applied across the other strands.

Book Nine: Teaching Number through Measurement, Geometry, Algebra and Statistics offers examples of how knowledge of the Number Framework can influence teachers' work with students on the different mathematics strands. The lessons provided are examples of how these links can be made and are not intended to be a comprehensive sequence or set. As with the other books in the numeracy series, a stage indicator has been used to suggest which strategy stages a given lesson is suitable for.

Book 9 is organised into the following sections:

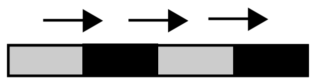
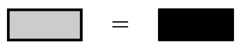
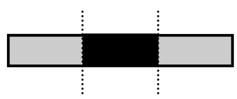
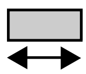
[3] Quoted in *Curriculum Update* 45 (February 2001), page 1.

Measurement Links to the Number Framework

In measurement, units are used to quantify the attributes of objects. These attributes are sometimes spatial, for example, length, area, volume; sometimes physical, for example, weight (mass) and temperature, and sometimes have no obvious physical connection with objects, for example, time. From these basic measures, rates are created, such as, speed, for example, km/h, and density, for example, g/cm^3. At a simple level, number connects to measurement because numbers are used to quantify the measurements, for example, 45 cm, 37.4°. However, the connections run much deeper than that.

Battista (2003)[4] lists four spatial processes necessary for students to understand the measurement of area and volume. These processes are imaging models, structuring models, locating units, and forming composites. The strong relationship between these spatial processes and quantifying can be summarised in the following developmental framework.

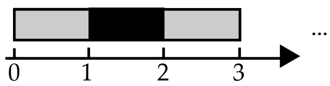
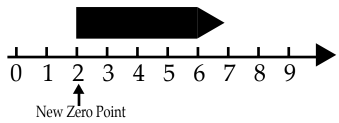
The framework provides conceptual stages in students' development in measuring length that can be applied across most attributes we might want to measure. It is derived from work by Lehrer, Jaslow, and Curtis (2003)[5] and Geary, Holton, Tagg, and Thomas (2003)[6].

	Idea	Description	Example
Concept of Attribute	Comparison	Objects can be compared in relation to an attribute they all possess.	A is longer than B A ——————— B —————
	Transitivity	A third object can be used to compare two other objects without needing to compare the objects directly.	A is longer than B, B is longer than C, so A is longer than C (without directly comparing them)
Concept of Unit	Iteration	A subdivision of an attribute is translated to obtain a measure.	
	Identical unit	Each subdivision is identical.	
	Tiling	Units fill a space with no gaps or overlaps.	
	Relativity	Units compare in size to other known objects (sense of size).	
	Partitioning	Units can be partitioned.	
	Additivity	Measures can be joined and separated, 8 units into 5 units and 3 units	 5 3

4 Battista, M. T. (2003). "Understanding students' thinking about area and volume measurement." In Clements, D. H. and Bright, G. B. (Eds.) *Learning and Teaching Measurement: 2003 NCTM Yearbook,* Reston, VA: NCTM.

5 Lehrer, R., Jaslow, L. and Curtis, C. L. (2003). "Developing an understanding of measurement in the elementary grades." In Clements, D. H. and Bright, G. B. (Eds.) *Learning and Teaching Measurement: 2003 NCTM Yearbook,* Reston, VA: NCTM.

6 Geary, M., Holton, D., Tagg A., and Thomas, G. (2003). *Assessment Exemplars: Mathematics,* Wellington: Learning Media.

	Idea	Description	Example
Concept of Scale	End points	The marks on a scale are put at the end points of the units.	
	Zero point	Any point can serve as the origin (start) of a scale.	
	Precision	The choice of units in relation to the object determines the relative precision of the measure. All measurements are inherently approximate.	

Piaget's early work suggested that students learn to conserve measurement attributes at the following ages: length, 6–7 years; area, 6–7 years; substance, 6–7 years; weight, 9–10 years; volume and capacity, 11–12 years. Kamii and Long (2003)[7] highlighted that most students begin to conserve speed (time) at ages 9–11 years.

While some researchers dispute the accuracy of these ages, it is clear that some measurement attributes are more difficult for students than others. For example, length and area appear to be easier than volume and capacity. These differences are due to the complexity of the spatial visualisation and / or quantifying involved.

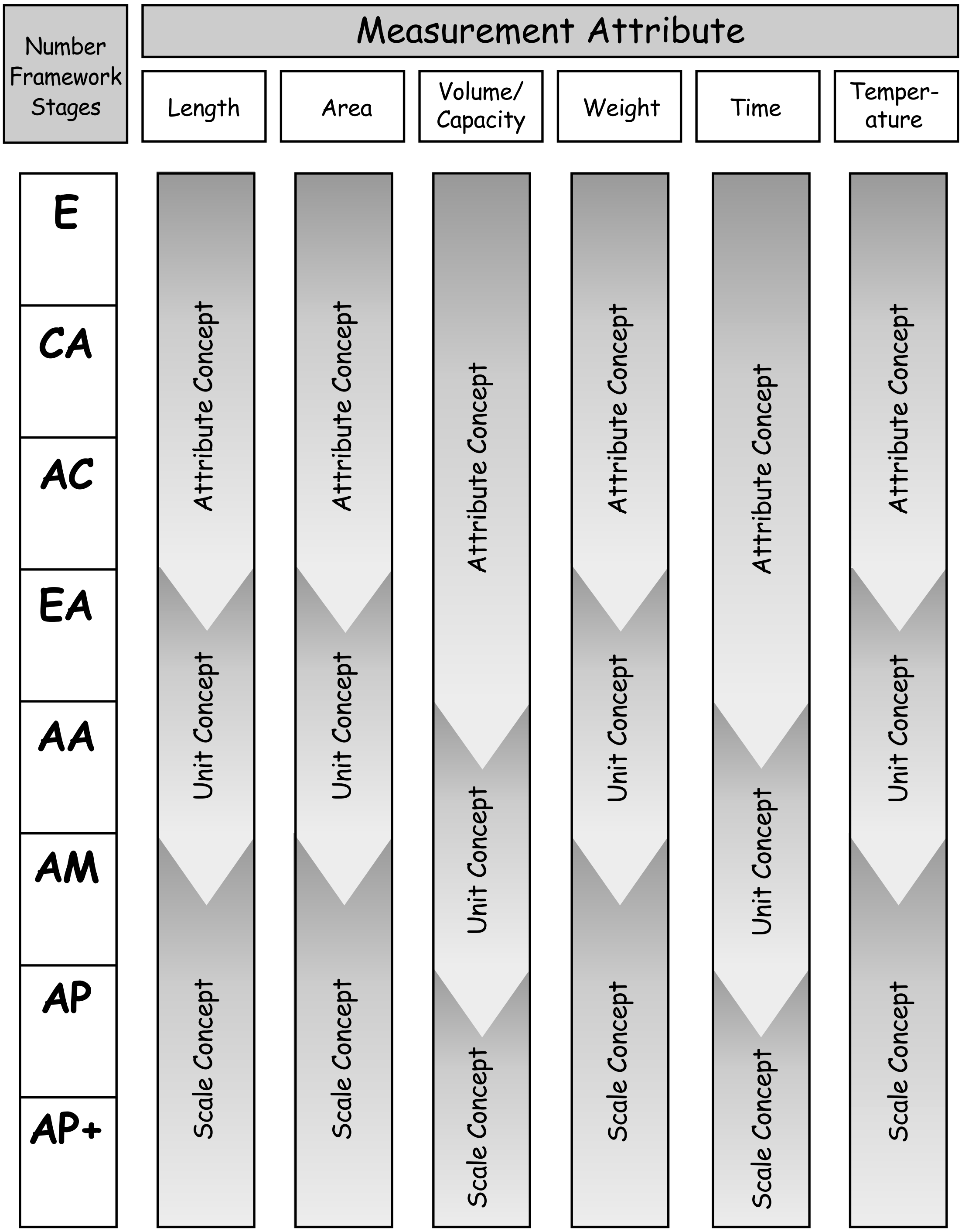
The measurement framework shows that there are clear cognitive connections between students' development of number strategies and their development of measurement ideas. Conception of a measurement unit parallels the development of both hierarchical (one more is the next number) and inclusive (the last count includes all of the others) counting. The unit concept is established when part-whole reasoning develops, for example, recognising that a measure of eight can be split into a measure of five and a measure of three.

The conception of scale marks the transition from discrete thinking, which is in units of one, to continuous thinking, which underpins the metric measurement system. Continuous thinking involves seeing a length, area, period of time, etc, as divisible into an infinite range of equal units. Smaller units are nested within bigger units, for example, millimetres are tenths of centimetres, which are hundredths of metres. The smaller the subdivisions (or units) chosen, the more accurate the measurement.

In the same way that the discrete unit concept is linked to counting, continuous unit thinking is linked to multiplicative reasoning. Understanding the metric system requires an understanding of decimals that emerges alongside strong multiplicative strategies for whole numbers. In some measures, particularly area and volume, the units are composites of ones, for example, rows, columns and layers. This suggests that students need multiplicative reasoning to understand some types of units and to be able to partition them.

7 Kamii, C. K. and Long, K. (2003). "The measurement of time: Transitivity, unit iteration, and conservation of speed." In Clements, D. H. and Bright, G. B. (Eds.) *Learning and Teaching Measurement: 2003 NCTM Yearbook*, Reston, VA: NCTM.

Connections to stages in the Number Framework are captured in the following diagram.



The key to the successful teaching of measurement lies in assisting students to apply the mental strategies they bring with them from number to situations that require strong spatial visualisation.

Learning Experiences to Develop Students' Reasoning in Measurement

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CA
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Putting Numbers on the Number Line

I am learning to find numbers on a number line or measurement scale.

Links to Other Strands

Number: representing operations on number lines

Statistics: reading and creating scales (axes) for graphs

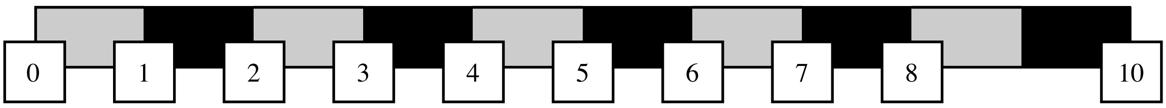
Required Knowledge

- Number sequences from 1 to 10 at least

Equipment: Number stick or a set of unilink cubes, a set of numbers that can be fastened (Material Master 9–1). (A number stick is a length of wood or plastic that is marked into 10 divisions using colour breaks.)

Using Materials

Show the students the number stick with 0 at the start and the other colour changes marked as 1, 2, 3 ... (but missing 9), using numeral cards fastened along the stick.



Ask, "What is the missing number?"

Repeat, using other missing numbers. Keep the same scale but progress to only showing the start numbers (0, 1, 2, 3). Ask what numbers will go on various other colour changes. Maintaining the unit scale, put 2, 4, 6, ... on every second colour change and ask what the missing numbers are.

Repeat with other sets of multiples, for example, fives, tens, threes.

Using Imaging

Show the students the number stick with 0, 1, 2, 3, ... labelling successive colour changes. Get them to draw their own number stick in their books. As a class, compare drawings and ask the students, "What is the simplest way to draw the number stick?" Discuss which features of the number stick are most important and how they can be drawn. If necessary, show how to make the representation a line. Highlight that it is straight and should be drawn with a ruler, that the numbers go at the marks and that the space between each of the numbers should be the same. Show the students the number stick with a variety of numbers along a unit scale and get them to draw what it would look like as a number line. The students should compare their drawings and give feedback to each other.

Using Properties of Scales

Ask the students to draw a number line and show unpatterned sets of numbers like 0, 2, 3, 6, 9, 11 or 4, 5, 8, 9 on it without any number stick model.

Extend this to scales with numbers beyond 10 and 20, and later to scales that do not have 0 as their starting point, for example, 25, 30, 35, ...

Learning to Measure Length

I am learning to measure how long things are.

Links to Other Strands

Number: fractional numbers, addition and subtraction strategies

Geometry: reflective and translational symmetry

Required Knowledge

- Counting and FNWS
- Bigger or smaller
- Students need to have had prior experiences of comparing different but similar lengths, for example, straws, and need to be able to answer the questions, "Which is longer?", "How can you tell?" and "Can you prove you are right?" This is aimed at getting students to attend to the attribute of length and to understand that direct comparison requires the lengths to be laid side by side with a common baseline or starting point.

Equipment: Pairs of straws cut to lengths of 13 and 15 centimetres, other straws cut to various lengths, paper strips made from adding machine tape, card or paper models of rulers marked in centimetres (Material Master 9–2).

Using Materials

Show the students a 15-centimetre-long straw and then cover it, then show them a 13-centimetre-long straw and ask, "Which straw is longer?" After the students have suggested answers, ask, "How can we tell which is longer if the two straws cannot be seen at the same time?" and discuss ways of doing this. If the idea of a unit of measure is not mentioned, suggest that the students can measure the straws with widths of their thumb and see how many thumbs long each straw will be.

After the students have measured their pairs of straws, discuss the use of thumb width as a unit of measure. Ask, "Why do some people get different numbers of thumbs to other people?" This will focus the students on the need for a standard unit of measure. Discuss the impact of leaving gaps between the thumbs, leading to the idea that they should be placed end on end with no gaps or overlaps.

Progress to measuring longer objects, such as the width of the room, with the students suggesting appropriate units of measure, such as arm spans, feet or paces. Record the students' measures on the board and discuss similarities and differences. Establish the need for a standard unit of measure.

Using Imaging

Establish a class unit, using the hand length of one student, then have each student cut straw copies of this unit to measure how long things are. The students can use this unit to measure a collection of objects and record their measurements. Discuss how to deal with lengths that are "a bit longer" than a whole number of standard measures (for example, 14 and a bit more hands). This leads to the idea that for increased accuracy, a new unit needs to be created that is a division of the standard unit.

Compare the consistency of the measurements to those taken previously with non-standard units. Suggest that every pupil cut a strip of paper that is exactly 10 hands long. Compare the strips, which should be the same length, and discuss any differences.

Ask the students how they could use this paper strip to speed up the measurement of objects within the classroom. This will lead to the students suggesting they create a "hands" ruler. Let the students make their "ruler" and practise using it. Ask the students how to use this "ruler", drawing out the need to line up the start of the ruler with one end of the object and to count the next mark as 1. Discuss the idea of numbering the marks on the "ruler" (if this has not already happened) and where 0 is located on a ruler.

Using Measurement Properties

Ask, "In the real world, how do we make sure that everyone in the world gets the same measurement?" Introduce the idea of the centimetre as a worldwide standard measurement. Give the students a single piece of straw that is 1 centimetre long and ask them how they are going to remember how long it is. Develop the students' mental image of a centimetre, then provide them with a paper ruler marked only in whole centimetres (Material Master 9–2). The students can use this ruler to measure the lengths of objects.

Have a similar discussion and progression for measuring large objects, like the room width, to establish the concept of the metre.

Get the students to cut an 8-centimetre-long straw and a 5-centimetre-long straw. Ask questions like, "How long will it be if I put these two straws end to end?", "How long will it be if I put six of these combined straws together end to end?"

Take another straw that is 16 centimetres long. Cut a 7-centimetre-long piece off the straw and tell the students that you have done this. Ask questions like, "How long will the leftover piece be?", "How long will it be if I cut off another 7 centimetres?"

These problems are designed to help the students realise that units can be separated and combined.

Related Activities

Introduce the millimetre as another standard unit of measure (for measuring small things) by focusing on the accuracy of measurements.

Get the students to measure their pencils in centimetres. This will raise issues of rounding as "closest to".

Choose pairs of students who have identical pencil measurements (such as, 14 centimetres) and get the students to directly compare their pencils. The students will note that centimetre measures were not accurate enough for comparing the lengths of pencils. Discuss what unit could be used to do this. This will lead to the idea of subdividing centimetres, which in turns leads to the idea of millimetres (tenths of centimetres).

Measure the pencils again using millimetres and compare the degree of accuracy with the centimetre measures.

Look at the relationships between metres, kilometres, centimetres and millimetres, noting the meaning of the prefixes, milli- (one thousandth), centi- (one hundredth), and kilo- (one thousand).

Using Multiple Scales Like 0, 5, 10, 15, ...

I am learning to find numbers on scales that show multiples.

Links to Other Strands

Number: number word sequences

Statistics: reading and creating graph scales

Geometry: reflective and translational symmetry

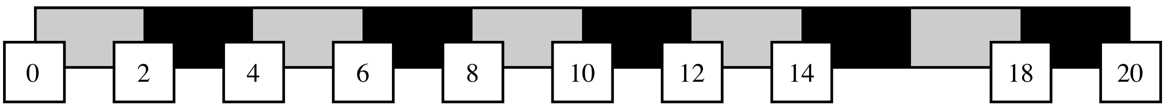
Required Knowledge

- Skip-counting in a variety of bases, for example, threes, fours, ...
- Halving numbers to 10, finding a number halfway between other numbers
- Rounding numbers to the nearest 10, 100 (stage 5)
- The students need to have successfully completed the activity Putting Numbers on the Number Line, page 6.

Equipment: A number stick and a set of numbers (Material Master 9–1).

Using Materials

Show the students the number stick with 0 at the start and 2, 4, 6 ... labelled on subsequent colour changes (but with 16 missing).



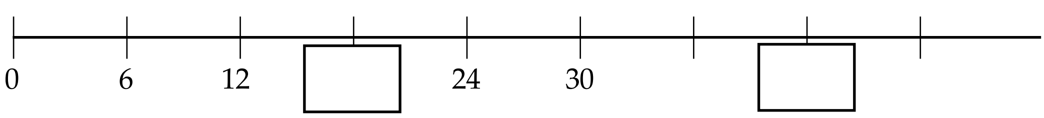
Ask, "What is the missing number?" Repeat with other missing numbers and other scales. Get the students to draw number lines to show what the number stick shows, then get them to give feedback on the quality of each others' number lines. Where necessary, revise the conventions discussed in Putting Numbers on the Number Line. It is worth making explicit the conventions that the numbers get bigger as you move to the right and that the intervals are consistent.

Using Imaging

Start again with just 0 and 2, 4, 6 on subsequent colour changes. Ask the students what numbers will go on various other colour changes. Repeat with other sets of multiples, for example, 0, 3, 6, 9, ... Get the students to draw number lines representing the number stick, along with a given missing number, for example, given the scale 0, 2, 4, 6, ..., get the students to show where 12 should go. Be aware that some students will place 12 at the next mark as they have not recognised that the intervals on a scale should always be consistent. If they make this mistake, explicitly tell the students about this convention. Use the number stick to confirm the placement is correct by either skip-counting or using multiplication if the students suggest it.

Using Properties of Scales

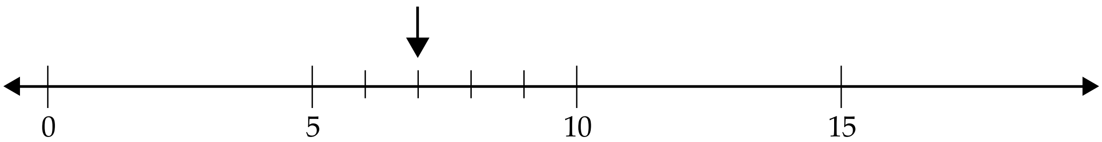
Have the students draw number lines with a given set of multiples as the scale and get them to practise continuing a scale and locating missing numbers on a scale. For example:



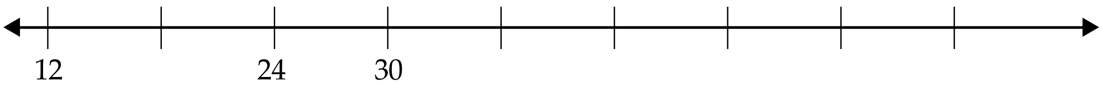
Related Activities

Develop the idea of subdividing intervals on a scale. For example, show the students the number stick with 0 at the start and 2, 4, 6, ... on every second colour change. Ask, "Where does 1 go?" Establish the language that "1 is halfway between 0 and 2". Repeat with 0, 2, 4, 6 on subsequent colour changes. Establish that 1 is still halfway between 0 and 2, even though there is no "mark" for it on the scale. Locate 3, 5, 7, ... as well. Repeat with multiples of 4 as the scale, finding 2, 6, 10, then progress to locating 1, 3, 5, etc.

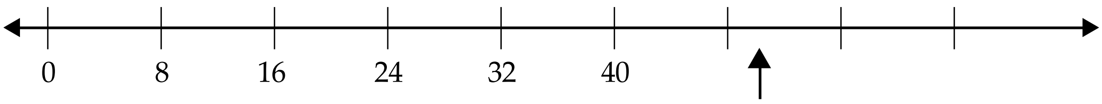
Use other multiple scales of increasing complexity and ask the students to locate "in between" numbers. Develop the students' ability to draw the scales as number lines. Discuss strategies for finding where certain numbers would be located with an emphasis on accuracy. For example, on a scale showing 0, 5, 10, 15, ..., 7 is located two intervals of one-fifth to the right of 5.



Once the students have a well developed control of zero-based scales, vary the starting number and intervals of the scales used. Have the students draw the scales and use arrows to locate numbers that are not shown. For example, "Find 26 or 35 on this scale."



Also pose questions that require the students to work out to which number an arrow is pointing to.



Relate the work on scales to reading scales in measurement contexts such as temperature (thermometers), weight (scales), capacity (buckets, jugs), and speed (car speedometers), etc.

Further Activities

Give the students a collection of numbers, such as, 6, 13, 26, 41. Tell them to come up with a scale they can use to show where all these numbers are on a number line. Have the students feed back their suggestions to the group and discuss which scales are the most appropriate and where the numbers would be found on the chosen scale.

Extend the range of scales progressively to other multiples like 12, 15, 20, 25, 50, 100, 200, 250, 500, 1000.

Use a rope number line laid out along the front of the class, with 0 pegged at one end and 1000 at the other. Have the students estimate where they think 639 would lie and peg a piece of paper with their name on it at that place on the line. Compare strategies for locating the number to reach a consensus on the number's location and on what was the best way of locating that number.

Investigating Area

I am learning to compare and measure areas.

Links to Other Strands

Number: addition and multiplication strategies

Geometry: tessellations of triangles, quadrilaterals, and other polygons
reflective, rotational, translation symmetry

Required Knowledge

- The student must have successfully developed an understanding of the concept of length and how to measure it.

Equipment: Paper rectangles, scissors, sticking tape, a collection of potential area units such as square tiles, beans, counters, pattern blocks.

Using Materials

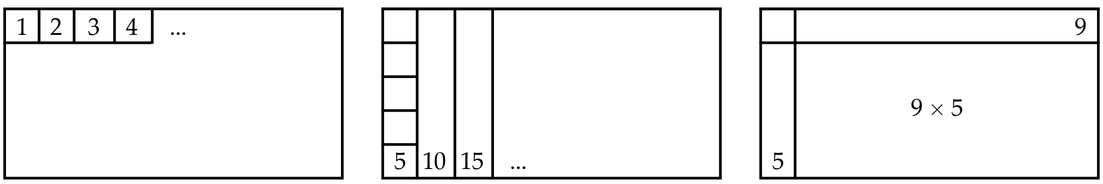
Show the students three different-shaped rectangles that have the same area and that can be directly mapped onto one another by cutting and moving the parts, for example, 12 cm × 12 cm, 6 cm × 24 cm, 8 cm × 18 cm.

This is primarily to get the students to attend to the attribute of area.

Label the rectangles A, B and C. Ask, "Which of these is the biggest?" Once the students' responses have been recorded on the board, ask, "How can we check which is the biggest?"

Have the students work in groups to develop strategies, then feedback to the class. Their suggestions might include direct comparison through visual "feel", overlaying, cutting and moving, measuring with a smaller rectangle (unit). Have them use their methods to compare and trial other sets of rectangles, for example, 10 cm × 10 cm, 3 cm × 33 cm, 22 cm × 5 cm. (Note that the rectangles suggested in the example above have slightly different areas. This is a good teaching point.)

All the students should be able to compare areas of rectangles. However, two aspects of using units of area need to be developed. Firstly, the students need experience in choosing, locating and combining units of area into rows or columns in order to cover a surface. Secondly, the students need to develop strategies for quantifying how many square units fill a rectangle. Here their number strategy stage will have a significant impact. This is summarised in the diagram below:



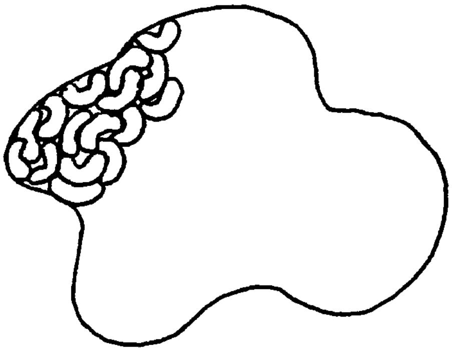
One to one counting

Advanced counting

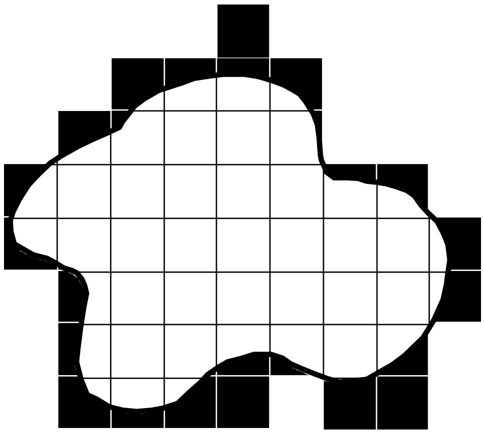
Early multiplicative (Advanced Additive)

After a couple of experiences comparing the areas of rectangles, the students may appear to see the significance of using a tessellating regular polygon as a unit of measure. Research indicates that students tend to choose a unit that matches the shape of the area they are finding, for example, they will measure a curvy shape with a curvy unit like beans. Get the students to trace around their feet and ask them to find out who has the biggest footprint.

Let the students choose the units they wish to use to solve this problem. After exploring the process of measuring with various units, discuss the implications for accuracy of using a unit that leads to gaps or overlaps (for example, counters). Discuss how to deal with units that are part inside and part outside the shape.



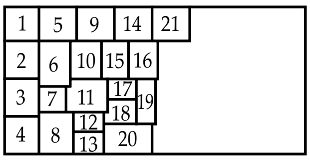
Using non-tessellating units creates inaccuracy through gaps and overlaps.



Using regular units creates accuracy issues through not fitting exactly.

Using Imaging

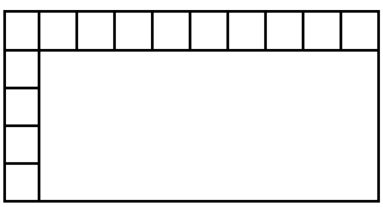
Unit location and the use of composites, that is, the way units are grouped, are important aspects in the students' ability to measure area. Provide the students with three rectangles and ask them to draw the internal square units in order to compare the areas of the shapes. Many students will have difficulty locating where squares would be placed and in maintaining squares of equal size.



Once the students can effectively image the location of units within an area, it is possible to focus them on grouping the units into rows or columns. There will be limits on how far Advanced Counting and Early Additive students can proceed in using rows and columns to find the total number of units. Multiplicative understanding of area occurs when the student is able to co-ordinate the number of rows with the number of columns. For example, a rectangle with 6 rows and 4 columns has an area of $6 \times 4 = 24$ units.

Using Measurement Properties

To proceed with these lesson steps, the students will need to have some multiplicative strategies. Show the students two objects with a similar area where direct comparison is very difficult, for example, 7 cm × 7 cm and 5 cm × 10 cm. Ask, "How can we find which area is bigger without overlaying these rectangles or filling all of the areas with squares?" To scaffold the students' reasoning, you could identify square units along two adjacent edges of the rectangles.



The students who can locate units by imaging will apply their number strategies to quantify the units and express their strategies in additive or multiplicative statements, for example, 8 + 8 = 16, 16 + 16 = 32 (Early Additive stage).

Pose problems that require using standard measures of area, particularly square metres (m^2) and square centimetres (cm^2). It is worthwhile making paper templates of these units, which can then be laid out along the edges of different rectangles, for example, room floors or exercise book covers, if necessary.

Many students have difficulty taking the next step of using a ruler to determine how many metres or centimetres are located along each side of a rectangle. This needs modelling by actually showing the students how the square units are laid out along the sides of a rectangle and then measuring with a ruler.

Fitting In

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I am learning to compare how much a box or container holds using a measurement unit.

Links to Other Strands

Geometry: tessellations, patterns of shapes that cover a surface with no gaps or overlaps (volume is a three-dimensional tessellation)
forming nets of simple polyhedra, such as, cubes and rectangular prisms

Algebra: number sequences, for example, predicting, "A box has a base of 10 cubes, how high must it be to hold 200 cubes?"

Statistics: data handling, for example, "How much of a cereal box is cereal and how much of it is air?"

Required Knowledge

- The students will need some experience with measurement of area (two dimensions) before proceeding to volume. This should involve using arrays as suggested in *Book 6: Teaching Multiplication and Division.*
- Experience with using a unit in simpler measurement contexts such as length is also a useful prerequisite.
- The students will also need some experience with using unconventional units such as cubes as well as conventional units, such as centimetres.

Equipment: A range of household boxes of similar volume but varying size, cubes (2 cm^3).

Using Materials

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Show the class three different-sized household boxes that have similar volumes (for example, cereal boxes, cracker boxes). This helps develop the idea that spatial perceptions, for example, taller boxes hold more, can be misleading.

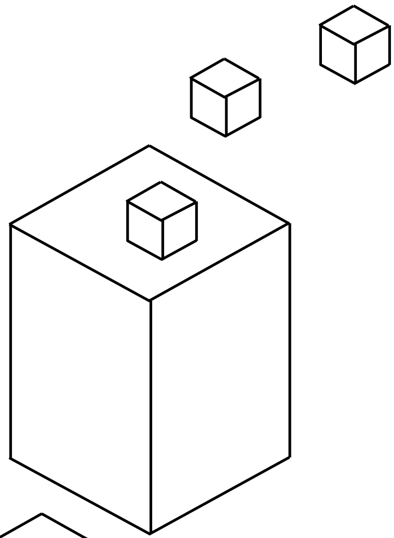
Tell the students that you want to find out which box holds the most cubes when they are packed in "tight". Ask, "Which box do you think holds the most cubes? Why?" Discuss ideas such as taller or shorter holds more and look at the more sophisticated ideas that volume involves length, width and height.

Divide the class into groups and, providing each group with a set of boxes and cubes, challenge them to find out which box holds the most cubes. Ask the students to record, using numbers, words and diagrams, how they solved the problem.

After a suitable period of investigation, bring the class back together to discuss their ideas. The students' methods will probably reflect their number strategies and spatial understanding, particularly their ability to image the location of cubic units.

The different methods used could include the following:

The student puts cubes in the box randomly with no attempt to pack them or fill the entire space. This shows a lack of realisation of the volume attribute or of a cube as a unit of measurement.

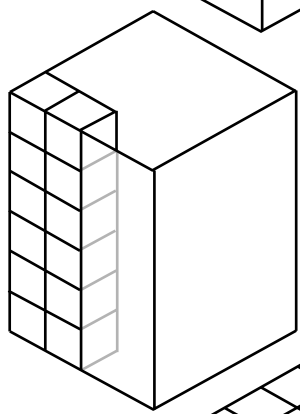


Count All

The student fills up the box with cubes one by one and counts the cubes one by one.
This shows recognition of the cube as a measurement unit.

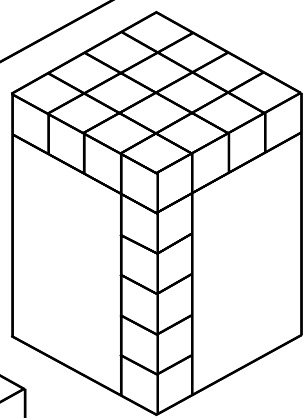
Advanced Counting/Early Additive

The student makes a column or row of cubes and finds out how many columns or rows fit in the box.
They are using skip-counting or addition facts to find the total number of cubes.



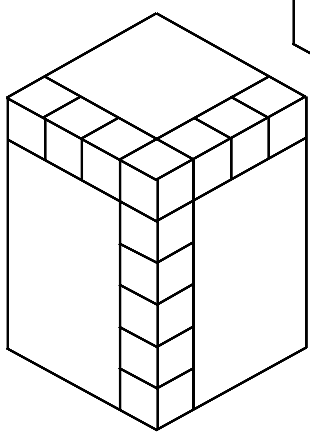
Advanced Additive and beyond

The student finds out how many cubes form a layer of the box.
They use the number of layers in the box to quantify how many cubes are needed.



Advanced Additive and beyond

The student finds out the number of cubes that fit along the length, width and depth of the box (three edges), and multiplies these numbers.



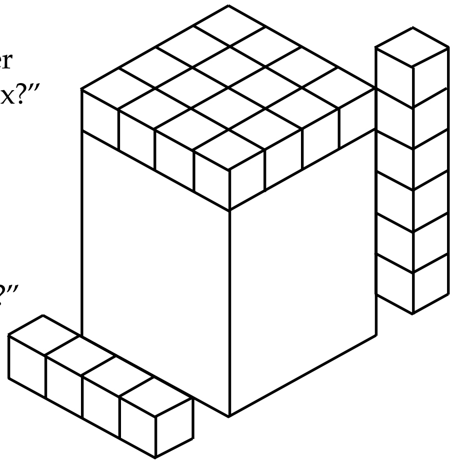
Avoid making evaluative judgments about the strategies the students use but ask questions about the efficiency of their methods. For example, "Which way takes the least work to do?" Be aware that the students may recognise the power of additive or multiplicative based strategies but lack the number knowledge to make these viable ways of quantifying the number of cubes.

Using Imaging

Show the students a different set of boxes. Tell them that they will need to decide which box holds the most cubes without filling the box up with cubes to check. Put the students into thinking groups of three or four to decide what information they would like to know that will help them work out which box holds the most cubes.

Invite the groups to share their ideas and record the information they want and, where possible, leave the physical model of the cubes in place. For example:

"How many cubes cover the top/bottom of the box?"

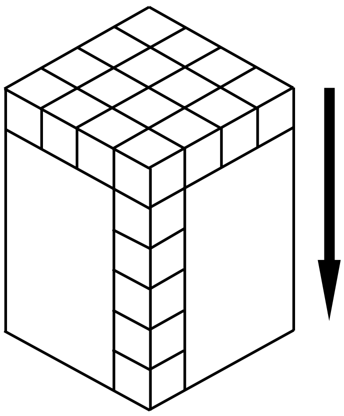


"How many cubes fit up the side of the box?"

"How many cubes fit along the side of the box?"

Ask the students to go back into their groups to decide which box holds the most cubes. After an appropriate time, bring the groups together and record the strategies that were used for each box. Use symbols and diagrams, for example:

Crunchie Crackers Box:



The student could solve this by adding each layer:
16 + 16 + 16 + 16 + 16 + 16 cubes = 96 cubes.

The student could use repeated addition to solve this:
10 + 10 + 10 + 10 + 10 + 10 = 60;
6 + 6 + 6 + 6 + 6 + 6 = 36;
60 + 36 = 96 cubes.

Promote efficiency in the strategies the students are using. This will involve creating links between counting, addition and multiplication.

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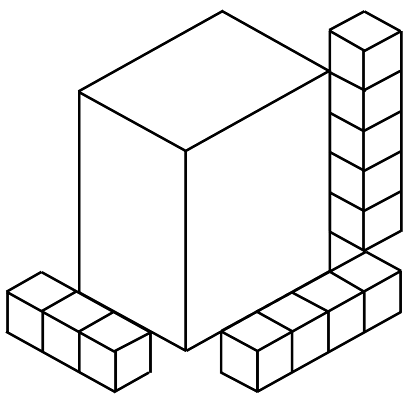
Using Measurement Properties

Problems such as those below should be used with students who have sufficient addition and/or multiplication strategies to cope.

Problem One
"Rachel has 36 cubes. She wants to make a box to hold them all.
How long could each edge of the box be?"
"Can she make many different-shaped boxes that will work?"
"Can she make a cube-shaped box to hold 36 cubes? Explain your answer."

Problem Two
"Here is a picture of Hemi's box.
How many cubes does Hemi's box hold?"
"Hemi wants to make a new box
that holds twice as many cubes.
How long should he make the edges
of the new box?"



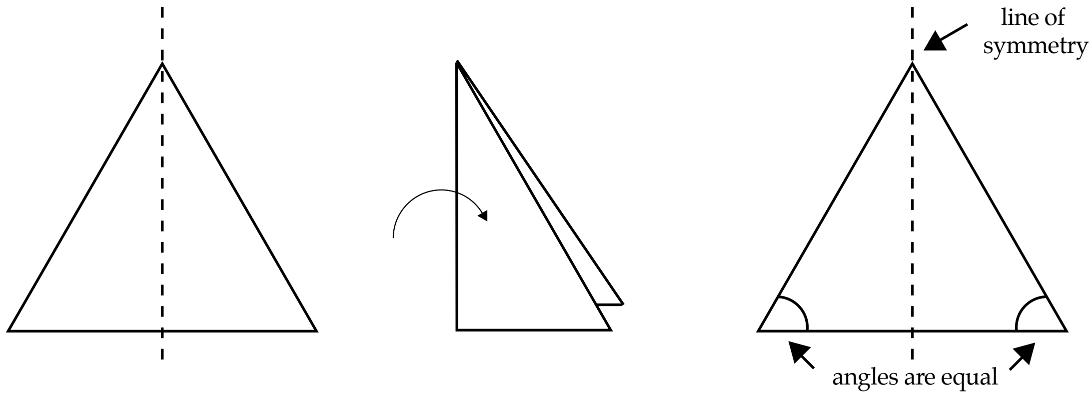
Geometry Links to the Number Framework

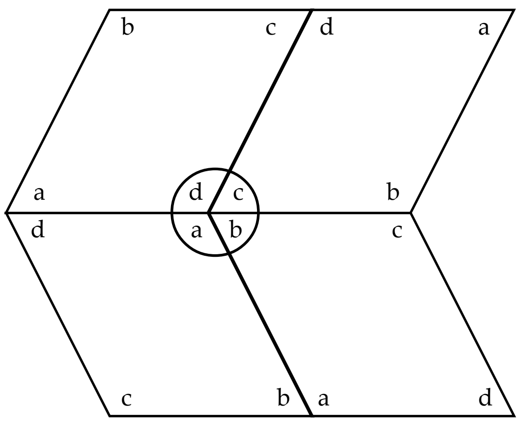
Links between spatial reasoning and the Number Framework are less obvious than for the other strands of the maths curriculum. This is partly because the progressions in spatial reasoning are less researched than those in number. Despite the lack of researched progressions, the importance of spatial reasoning is unquestionable. High correlations have been found between students' spatial ability and their general mathematical achievement.

Piaget and Inhelder (1967)[8] argued that students did not learn geometry simply by reading their environment, rather they learnt it by acting on and manipulating their environment. Clements and Battista (1992)[9] suggested that geometric thinking developed through students using both measurement and transformation to act on objects.

The quantifying measurement links to number have already been discussed in this book. Transformation means processes like reflection and rotation, partitioning and moving, in which a shape is mapped onto itself or other shapes. These transformations are an important way of establishing similarity and difference within and between shapes.

For example, folding an isosceles triangle in half will establish its line of symmetry and where interior angles are equal. Manipulating quadrilaterals through rotation, reflection and translation so that they tessellate helps students to establish the conditions under which all quadrilaterals tessellate and to recognise the sum of interior angles as a full turn (360°).





[8] Piaget, J. and Inhelder, B. (1967) *The child's conception of space.* New York: W. W. Norton & Co.

[9] Clements, D. H. and Battista, M. T. (1992) "Geometry and Spatial Reasoning". In D. A. Grouws (Ed.) *Handbook for Research on Mathematics Teaching and Learning*. Reston, VA: National Council of Teachers of Mathematics, pages 420–464.

One of the most accepted progressions in geometric reasoning was developed by Pierre and Dina van Hiele[10]. They described five increasingly sophisticated levels of thinking about shapes and space. In the framework below, an even more basic level has been added. This was not part of the van Hieles' original theory.

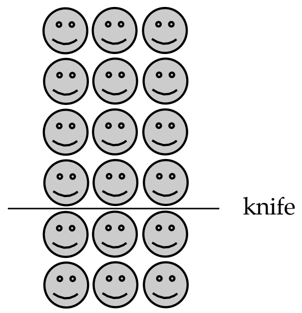
Level	**Title**	**Description**
0	Pre-Recognition	The student is unable to identify and form images of many common shapes, and recognises only a few of a shape's characteristics when classifying it, e.g., he/she recognises curved or straight sides but cannot distinguish triangles from squares.
1	Visual	The student recognises shapes by visual comparison with other similar shapes rather than by identifying the properties of the shapes.
2	Descriptive/Analytic	The student classifies shapes by their properties, e.g., squares by four equal sides, four right angles, but does see relationships between groupings of shapes, e.g., squares as kinds of rectangles.
3	Abstract/Relational	The student classifies shapes hierarchically by their properties, i.e., accepts classes within classes, e.g., squares are special cases of rhombi. He/she deduces that one property implies another and reduces definitions to their simplest form, e.g., a quadrilateral with pairs of parallel sides must have opposite sides of equal length.
4	Formal Deduction	The student is able to operate logically on statements about geometric shapes, solve problems and prove new results from these statements.
5	Rigor/Mathematical	The student is able to construct formal mathematical systems that elaborate on and compare relationships without reference to models of the geometric shapes involved.

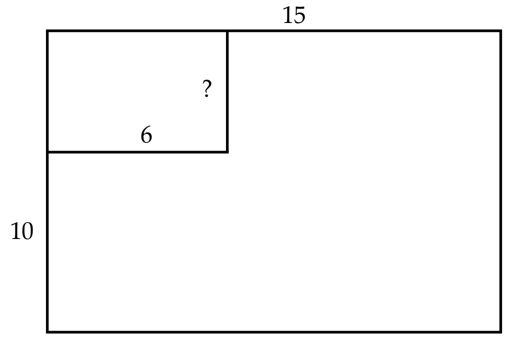
For students within the schooling system, levels 0–4 of the above framework of thinking are the most relevant. In fact, most researchers agree that the goal of secondary school geometry instruction should be to have students achieving level 2 and 3 reasoning (Clements and Battista, 1992). Early geometry instruction should be devoted to exploring and classifying shapes through measurement and transformation.

In applying the van Hiele levels, it is important to recognise generally that reasoning with two-dimensional shapes, like squares and circles, is easier than reasoning with three-dimensional shapes, like cubes and spheres.

[10] Van Hiele, P. M. (1959/1985) "The child's thought and geometry". In D. Fuys, D. Geddes, and R. Tischler (Eds.) *English Translations of Selected Writings of Dina van Hiele-Geldof and Pierre van Hiele.* Brooklyn, NY: Brooklyn College, School of Education. (ERIC Document Reproduction Service No. 289 697), pages 243–252.

The introduction of this book suggests that a teaching model that emphasises high quality imaging of materials and actions on materials is likely to develop students' spatial reasoning. There is also an extent to which spatial reasoning helps inform students' number thinking. Two examples are given below:





In both cases, the spatial representation informs the numeric reasoning. The knife problem explicitly shows how one of the factors in a multiplication can be distributed. The proportional reasoning problem, involving two overlapping similar (in shape not size) rectangles, gives reference to the scale factor that has been used to map one rectangle onto the other. Compare these images with the equivalent numeric representations: $6 \times 3 = (4 \times 3) + (2 \times 3)$ and $10 : 15 = \square : 6$?

Learning Experiences to Develop Students' Reasoning in Geometry

This section develops lesson scenarios in which both measurement and transformation are used to help students develop geometric reasoning.

Constructing Shapes Using Triangles

I am learning to construct and name shapes made by joining triangles.

Links to Other Strands

Fractions (including equivalent fractions)

Using a systematic approach to count all possible outcomes (probability and mathematical processes)

Required Knowledge

- Read and model the fractions, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{6}$, ...
- Name simple polygons, for example, "triangle", "square", "rectangle"
- Describe shapes using the language of geometry – "sides", "corners", etc
- Know vocabulary – "regular polygon", "equilateral triangle"

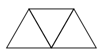
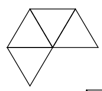
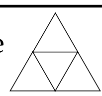
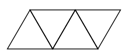
Equipment: Geometric shapes, such as pattern blocks or attribute blocks.

Using Materials

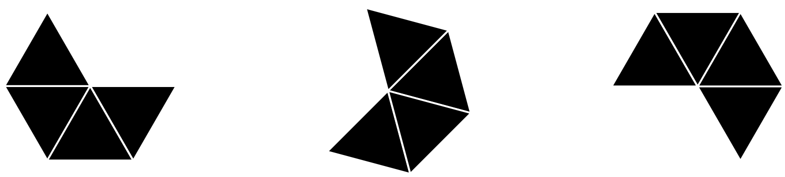
Select the equilateral triangle from the set of shapes. Tell the students that they are going to explore what polygons can be made using the triangle as a base.

"Using three equilateral triangles, how many different shapes can you make?" (Each triangle must share a side with another triangle, rather than a point.) "Name the shape as you make it." (For example, "trapezium".) "Now using four triangles, see how many different shapes you can make."

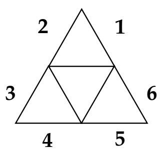
Get the students to record their solutions in a table like the one shown below.

Number of triangles used	**Number of different shapes made**	**Shapes made and pictures of those shapes**
3	1	trapezium 
4	3	hexagon  triangle  parallelogram 

You will need to discuss with the students the meaning of "different" in this case. Shapes will be considered different if they do not map onto themselves by reflection or rotation. For example, the following shapes may appear different, but they are actually reflections and rotations of one another.

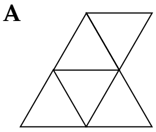


Once they have found all the possible shapes for four triangles, ask the students to predict how many shapes can be made using five and then six triangles. Ask the students if there is a pattern in the number of shapes that can be made. Encourage them to approach the problems systematically. For example, to find the different five triangle shapes they could consider all the four triangle shapes and where the extra triangle could be put to form a new shape.

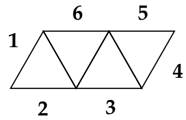


4 triangle shape

A

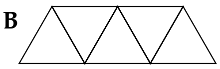


Joining the extra triangle to any side gives this shape.



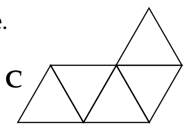
4 triangle shape

B



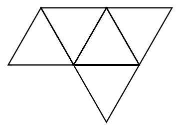
Joining to sides 1 or 4 gives this shape.

C

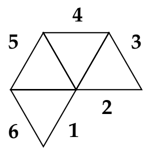


Joining to sides 2 or 5 gives this shape.

D

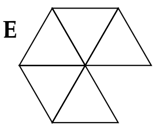


Joining to sides 3 or 6 gives this shape, which is the same as **A** above



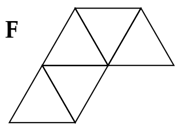
4 triangle shape

E



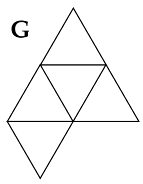
Joining to sides 1 or 2 gives this shape.

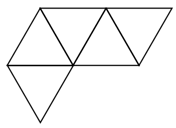
F



Joining to sides 3 or 6 gives this shape, which is the same as **C** above.

G

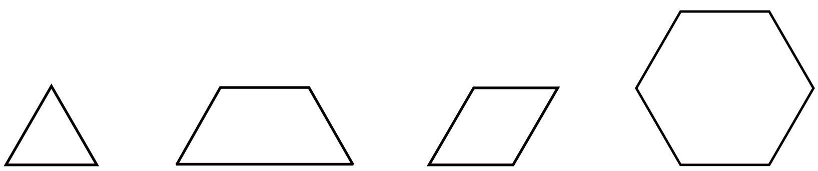




Joining to side 4 gives **A**, to side 3 gives **C**.

Using Imaging

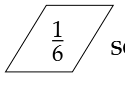
Using a set of pattern blocks, investigate the fractional relationships between the shapes. Give the students the shapes below and ask them to imagine whether each shape can be cut into congruent (identical) halves, thirds, quarters, fifths or sixths. Ask the students to explain why or why not the partitioning will work. Their explanations should involve ideas of symmetry, particularly reflective symmetry.



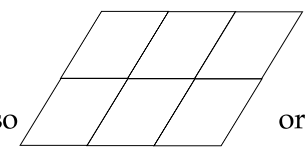
It is good to repeat the fractioning task with other shapes, for example, pentagons and octagons, including shapes that have more limited symmetry. This will help the students to generalise the process of finding whether a shape can be partitioned into given fractions.

For example, to divide a shape into sixths requires the shape to be halved, then each half divided into thirds.

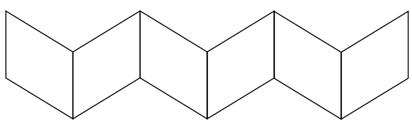
Reverse the problems to develop part-to-whole fractional reasoning. For example: "If this piece (below) is one-sixth of a shape, what was the shape?"



so



or



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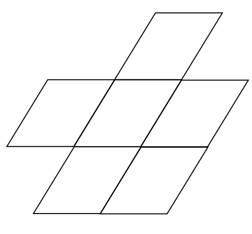


Be aware that these types of problems have multiple solutions.

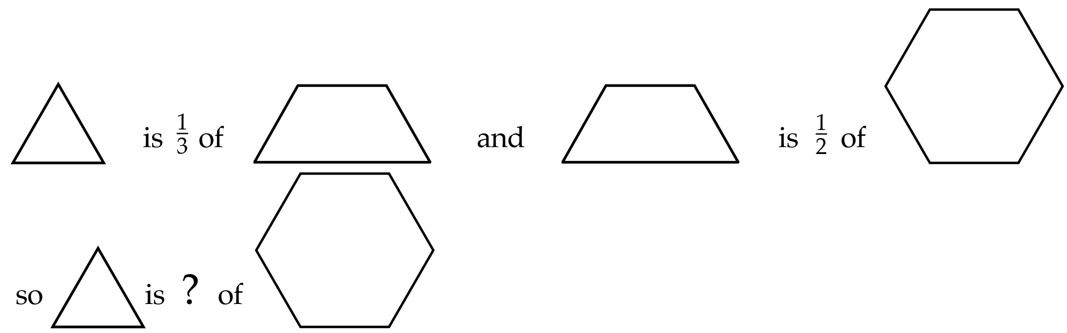
Using Number and Geometry Properties

Use the shapes to create fractions for fraction problems. For example:

"What fraction of the trapezium is the triangle?"

"What fraction of the hexagon is the trapezium?"

"If you know these two things, what fraction of the hexagon must a triangle be?"



Create other larger shapes that the students are less familiar with so that they have to apply number properties to solve the problems. For example:

"What fraction is the triangle of the trapezium?"

"What fraction is the trapezium of the parallelogram?"

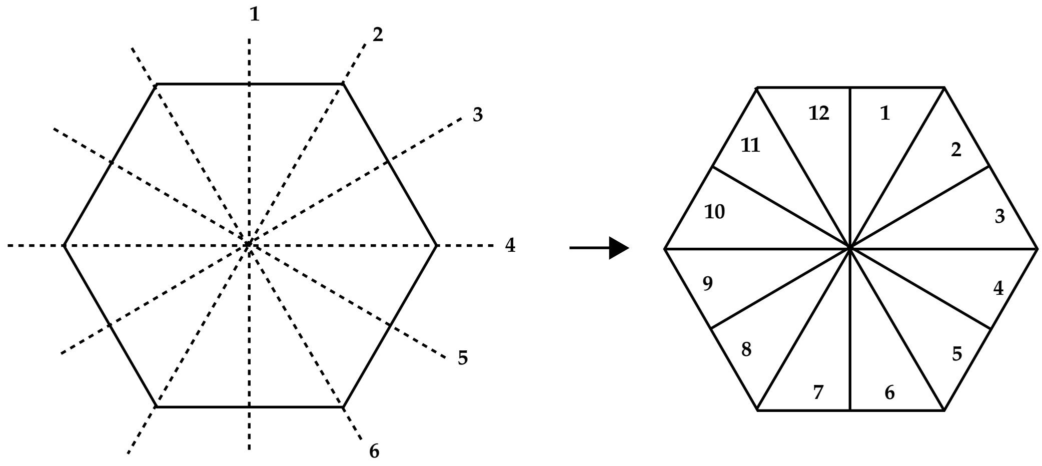
"Therefore, what fraction is the triangle of the parallelogram?"

Note that this last can be written as $\frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$ since the triangle is one-third of the trapezium, which is one-quarter of the parallelogram.

Related Activities

Take a shape, for example, a trapezium, and ask, "If this shape were 20% of another shape that I have hidden, what could my hidden shape look like?" "What shape(s) can be made? How many different solutions are there? How do you know that you have got them all?"

Investigate the link between the lines of symmetry a shape has and what congruent fraction pieces it can be cut into. For example, "A regular hexagon has six lines of symmetry and can be cut into halves, thirds, quarters, sixths and twelfths but not into fifths, eighths or ninths. Why is this?" Note that six lines of symmetry divide the hexagon into 12 parts, so the possible fractions have denominators that are factors of 12.



By the same reasoning, a regular pentagon has five lines of symmetry that divide the figure up into 10 equal pieces. Therefore a regular pentagon can be divided into halves, fifths and tenths, since 2, 5, and 10 are the factors of 10.

Angle Detector

I am learning to understand angles by relating them to turns, and I can measure angles in degrees.

Links to Other Strands

Number: fractional numbers
multiplicative and proportional reasoning

Measurement: interpreting scales

Required Knowledge

- Identify fractions of shapes
- Clockwise, anticlockwise rotations
- Addition strategies with numbers to 1000

Equipment: “PacMan” Material Master 8–7, protractors, measuring tape, unilink cubes.

Using Materials

The angle estimator (or “PacMan”) is a useful piece of equipment to use to explore fractions. It is a regions model. The students can show you fractions like one-quarter. Remember to ask about the other colour on the “PacMan” and what fraction that represents: “So if one fraction is one-quarter, the other part is what?”(three-quarters) Note that in a full turn the fractions add to 1.

Have the students work in small groups to discuss what an angle is. Get each group to share their ideas. The students may have four different ways of viewing an angle:

- The area between two lines (or space between the lines)
- The corner of a two-dimensional figure or where the lines intersect
- The two lines
- An amount of turn (This view underpins the others as it leads to degrees and standard units.)

The concept of an angle needs to be introduced as an amount of a rotation (turn). The following activity will help the students realise that the area between the lines is not the relevant feature when comparing angles.

Ensure that a variety of "PacMan"s with different circle sizes are available.

Ask the students to show one-third on the "PacMan". In groups, discuss what is the same and what is different about what is shown on the "PacMan"s of different sizes. List the similarities and differences:

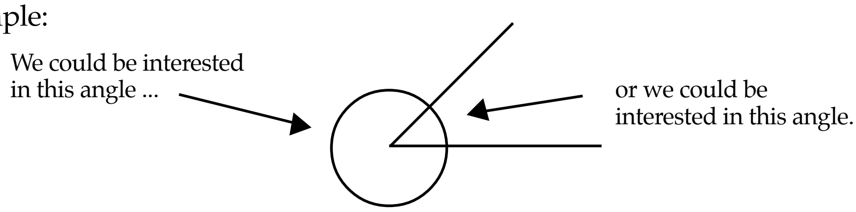
Same	**Different**
Fraction	Area
Two lines meet	Length of the lines
Turn	Size of circle

Repeat this for other fractions, such as three-quarters and two-fifths, to establish that the similarities and differences are independent of the fraction. It is through anticipating the action that the students will recognise an angle is about the amount of turn.

Ask the students to form an angle on their "PacMan". Ask, "What will you need to do to make the smallest angle bigger?" (rotate one circle). Model this with two rulers to show that turning increases the angle. After a few experiences with increasing and reducing the angle, list the features of a "PacMan" that change and stay constant under turning:

Change	**Stay Constant**
Fraction	Point where lines meet
Area between lines	Size of circle
Angle	Length of lines

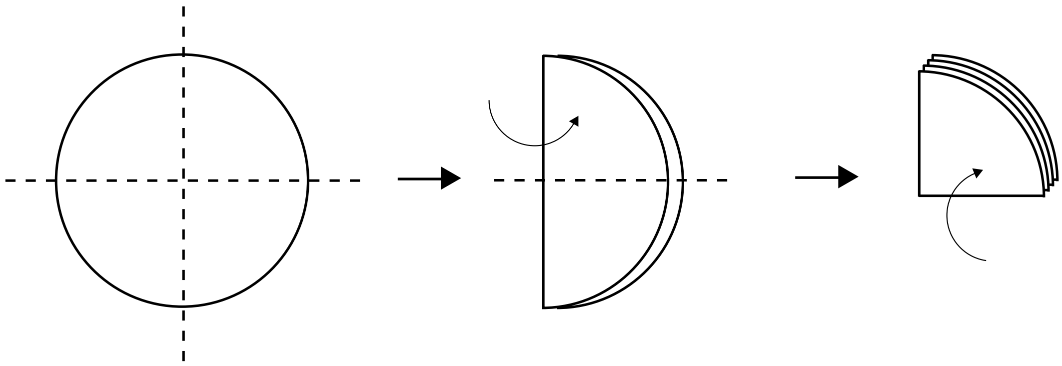
Link "PacMan" turns with body turns. Have the students face a particular point in the direction of North (either in the classroom or outside) as a reference direction. Ask questions like, "If you did a quarter turn this way (clockwise), what other turn could you do that would finish with you facing the same spot?"(that is, a three-quarter turn anticlockwise). Have the students replicate the turns using their "PacMan". Try other clockwise and anticlockwise turns, such as two-thirds and one-fifth. This activity also focuses on the idea that there are two angles (turns) that make up a full turn. The "PacMan" also demonstrates this with the two colours showing two fractions. For example:



Move into looking at turns greater than one rotation if the students show a clear understanding of the first part of the activity. For example, make a $1\frac{1}{2}$ turn clockwise.

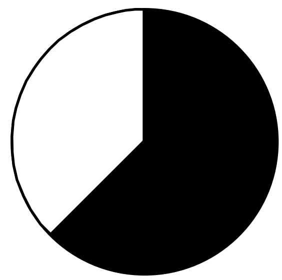
Get the students to create an angle tester by folding a circle into quarters and marking the creases. Ask questions like, "If I am standing at the centre of the circle facing North, what turn would I need to make in order to face the next line on my circle?" (a quarter turn).

Say that the angle that has been made has a special name, a "right angle". Fold the circle so that only one quarter is showing and you have a right angle.



This quarter circle can then be used as an angle tester. It tests whether an angle is greater than, less than or equal to a right angle. The students can go around the room classifying items into these three groups, using their angle tester. Discuss why right angles are so common in buildings.

Using Imaging



Show the students an angle on the "PacMan" that can not be measured in an exact number of right angles, for example, 135°. Ask, "How many right angles would fit in each angle of the 'PacMan'?"

The students may suggest that more divisions are required to accurately determine the angle size, with answers like, "It's one and a half right angles."

The students might then fold their angle tester in half again, creating eight divisions.

Invite them to name the new unit. It could simply be called "half a right angle", or preferably an "eighth turn" since eight turns like that form a full turn. The students could then try and find items in the room that have a one-eighth angle in them.

Ask the students to draw angles that could not be measured exactly using the one-eighth angle. Ask, "How small will the angle tester need to be to measure these angles?" The key idea is that increasingly small partitions result in more accuracy but may still not mean that the angle could be measured exactly.

Once the students have explored this idea, they are ready to be introduced to the standard unit of the degree.

Using Geometric Properties

Give the students a protractor and ask them to work out how angles could be measured using the device. Discuss some benchmark angle measures such as the right angle, which measures 90°. Get the students to use this information to work out the measure of a half turn (180°), full turn (360°) and one-eighth turn (45°).

The students may use a variety of mental number strategies to solve such problems. For example:

To find a half turn and full turn, measure:

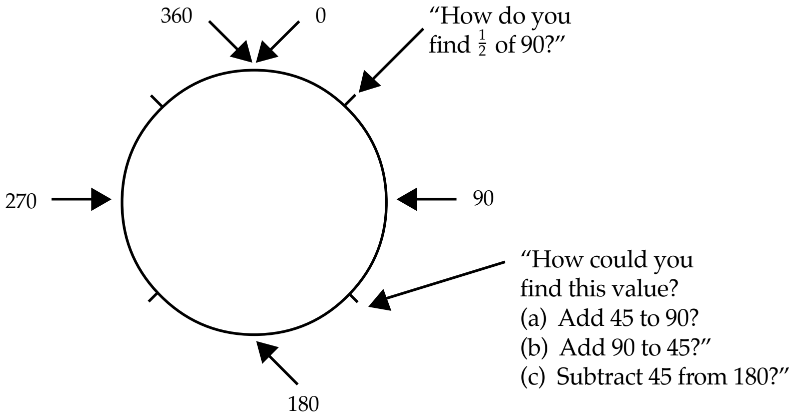
$(90 + 10) + 80 = 180$, $180 + 180 = 400 - 40 = 360$ (Early Additive and Advanced Additive stages).

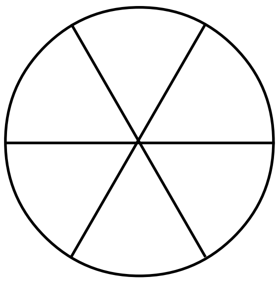
To find an eighth turn, measure: $90 \div 2 = 45$ because $45 + 45 = 90$.

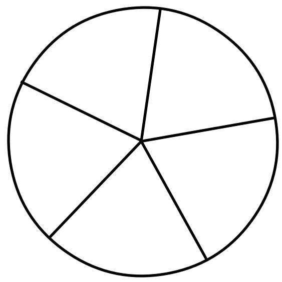
The students may like to investigate the origin of this division of one turn into 360° from the history of ancient Babylonia.

Pose more challenging problems that require the use of multiplicative reasoning. For example, "What is the measure of a one-third turn, a one-sixth turn, a two-fifths turn?" Encourage the students to use their mental strategies, for example, $360 \div 5 = (360 \div 10) \times 2 = 72°$ (Advanced Multiplicative stage). The students can check their calculations by forming the angles on their "PacMan", using a protractor, and visually checking their accuracy. This will involve ideas of fractions and symmetry. For example, a one-sixth angle can be checked by realising that three one-sixth angles form a half turn.

Give the students a paper circle and tell them to create their own simple protractor with key benchmark angles marked on it. Focus on discussing the mental calculation strategies for working out the measure of each benchmark angle.







The students might use their knowledge of 180° to calculate the angle measure of a one-sixth turn, that is, 180 ÷ 3. Strategies to solve this could include:

(a) (3 × 50) + (3 × 10) = 150 + 30, so the angle is 50 + 10 = 60°

(b) 3 × 6 = 18 so 3 × 60 = 180°

The fifths measure is more difficult for the students to calculate as there is not a known angle that the students can use. Strategies may include:

(a) (5 × 80) = 400, 400 – 40 = 360 so each angle is 72°

(b) (5 × 60) = 300, 5 × 10 = 50, 5 × 2 = 10, so each angle is 60 + 10 + 2 = 72°

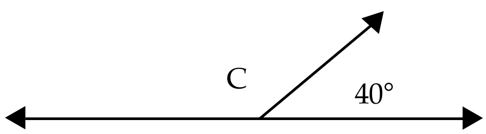
For the students who have a good number knowledge, extend this sequence of questions to angles past 360°. For example:

"Describe a 540° turn in a different way."

"If an object is rotated 1080° by turning the object three times, what three turns could have been made?"

Related Activities

Discuss the "straight angle" (180°). Link the angles between 0 and 180 to the angle sum of a straight line.



"If one angle is 40°, how many degrees is angle C?"

Investigating Polygons

I am learning to find the sum of the interior angles of a polygon and each interior angle of a regular polygon.

Links to Other Strands

Algebra: recognising the rule for a pattern and generalising; interpreting formula

Number: using multiplicative strategies

Required Knowledge

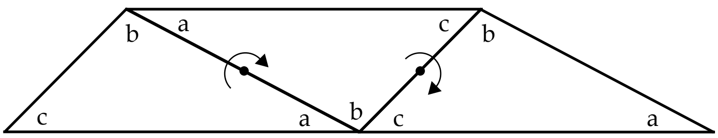
- Recall multiplication facts to 10 × 10 and the corresponding division facts
- Vocabulary – "polygon", "regular polygon", "equiangular", "equilateral"

Equipment: Polygon shapes, for example, pattern or attribute blocks, protractors.

Using Number Properties

The students would need to have completed the activity "Inside Out", Figure It Out, *Geometry*, levels 3–4, page 5 before this lesson. This activity explores the angle sum of triangles and rectangles through hands-on exercises, with the students ripping off the corners of a triangle and rearranging the pieces to form a straight line. Students who do not have the concept of an angle as a turn will struggle to see the relationship between the corners meeting and the angle sum of 180°, also known as a half turn.

The students might also tessellate with equilateral triangles and notice that three interior angles make a half turn. This generalisation is more difficult for other types of triangle since rotation about the mid-points of sides is required for triangles to tessellate in general. The students might investigate this with cardboard copies of scalene triangles. They may note that for the tessellation to work, each angle must be used around a point twice.



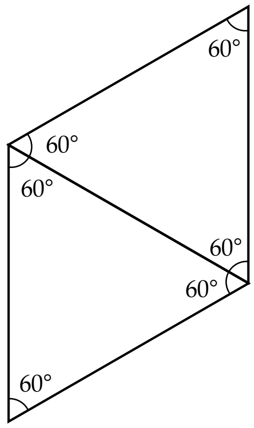
Another approach to help the students "discover" the angle sum of a triangle is to have them draw a variety of different-shaped triangles and measure the angles using a protractor. After completing a few examples, the students can share their answers with the class along with their drawings. This will lead to the discovery that the angle sum of different triangles are roughly the same. Variances can be discussed (accuracy of measuring).

Ask the students, "If we added up all the angles in a triangle, what would be the total?" If the students seem unable to work out the answer, revisit the activities mentioned at the start of this lesson.

Explore the interior angle sum of different polygons using the students' knowledge of the angle sum of a triangle. It might be useful to record the students' results in a table. For example:

Number of sides	3	4	5	6	7	...
Number of triangles	1	2	3	4	5	...
Interior angle sum	180	360	540	720	900	...

Put two triangles together (as shown).
"What shape do we have here?
What is the sum of the interior angles of the quadrilateral?"



The students may use a variety of strategies to solve the problem. Possible solutions may be:

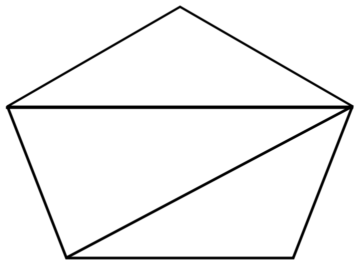
(a) $6 \times 60 = 360$

(b) $180 + 180$ as $200 + 200 = 400$, $400 - 40 = 360$

(c) $120 + 120 + 120 = 360$

The final solution could be verified by marking the corners of any quadrilateral, tearing them off and placing them together around a point.

"What happens if you use a different-shaped triangle as your base? Does this affect the original answer?" (No)



"Make a pentagon using any shaped triangle as a base.

What is the sum of the interior angles of your pentagon?"

Encourage the students to use their mental number strategies to solve angle sum problems. Possible solutions for the pentagon might be:

(a) $360 + 180$ or $400 + 140 = 540$ (by taking 40 from 180 and adding it to 360 to make 400)

(b) $360 + 180$ or $400 + 140$ (by adding the hundreds together and then the tens)

(c) $360 + 180 = (4 \times 90) + (2 \times 90) = 6 \times 90 = 540$ (common factors of 90°)

"Is there a way to work out the interior angle sum for a dodecagon? (12-sided figure) Are you able to use the other results to help you do this?"

The students may notice that adding an extra side to the polygon is equivalent to adding another triangle. For example, this means that the sum of the interior angles of a hexagon is 180° more than that of a pentagon. The students could use this "adding on" idea to extend their table of interior angle sums for shapes until they find the angle sum for a dodecagon.

Number of sides	3	4	5	6	7	8	9	10	11	12
Number of triangles	1	2	3	4	5	6	7	8	9	10
Interior angle sum	180	360	540	720	900	1080	1260	1440	1620	1800

The students with strong multiplicative strategies may look for a predictive rule that links the number of sides with the angle sum. These students may express their findings in word rules, for example, "Take two off the number of sides and multiply by 180 to get the sum of the angles." This can be written algebraically as $s = 180\,(n - 2)$, where n is the number of sides and s is the angle sum in degrees.

Ask the students to use the information from the angle sum table to draw a triangle where all the angles are equal (an equilateral triangle). Encourage them to discuss in groups how they solved this problem. Their solutions will involve division, that is, $180 \div 3$, to find the measure of each interior angle in degrees.

Then ask the students to draw a square and a regular hexagon. Have them share their strategies for solving how to draw these shapes, in particular their division strategies. Ask the students to think about what a regular pentagon looks like and to draw it on their desk with their finger. Ask them to estimate the size of each angle in degrees.

"What do you notice about the angle size as you progress from the triangle to square to pentagon?" Have the students extend their table to include a row headed "Interior angle of regular polygon".

Number of sides	3	4	5	6	7	8	9	10	11	12
Number of triangles	1	2	3	4	5	6	7	8	9	10
Interior angle sum	180	360	540	720	900	1080	1260	1440	1620	1800
Interior angle of regular polygon	60°	90°	108°	120°	$128\frac{4}{7}$°	135°	140°	144°	$147\frac{3}{11}$°	150°

Pose problems like, "Draw a regular polygon with 20 sides of 5 centimetres," and ask questions like, "Why can't the interior angles of any regular polygon, no matter how many sides it has, be greater than 180°?"

Related Activities

The following activities use regular polygons to explore tessellation and rotational symmetry. The shapes need to be available for the students to manipulate.

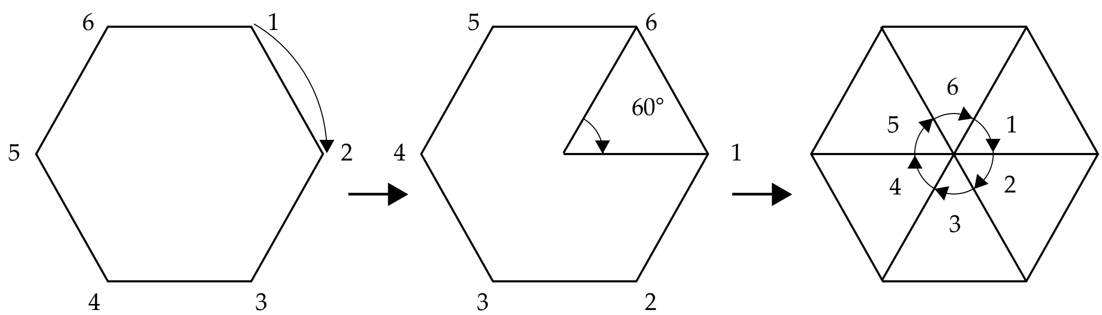
(a) "Choose one regular polygon. Does this polygon tessellate by itself? Which other polygons will tessellate by themselves? Such tessellations are called regular tessellations." Refer the students to the activity "Tiling", Figure It Out, *Geometry,* levels 3–4, page 6. Discuss why these tessellations work (side lengths are the same and the angles meeting at each vertex total 360°).

(b) See also the Assessment Exemplar for Mathematics, Geometry (hard copies are available in school or through TKI).

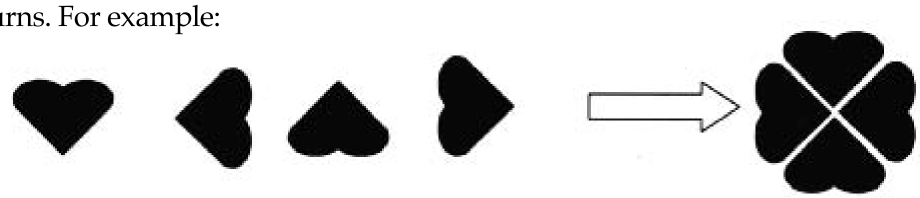
(c) "Which different regular polygons will tessellate together?" For example, "Do equilateral triangles and squares tessellate with each other? How many combinations are there using two different regular polygons each time?"

(d) "Can you tessellate using more than two different regular polygons? Why/Why not?"

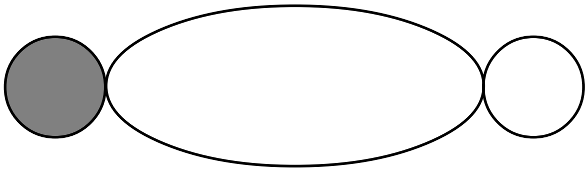
Explore rotational symmetry with the regular polygons. Classify shapes according to their order of rotational symmetry and the number of lines of reflectional symmetry. For example, a regular hexagon has six lines of reflective symmetry and rotational symmetry of order 6. This means that it maps onto itself six times in a full turn.



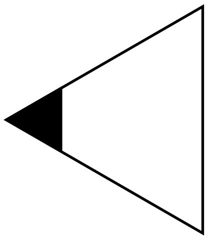
(a) Tell the students to create shapes/patterns that have certain types of symmetry. For example, "Make a shape that has rotational symmetry of order 4." Computer software can be used to create an element, copy it, and rotate the copies by quarter turns. For example:



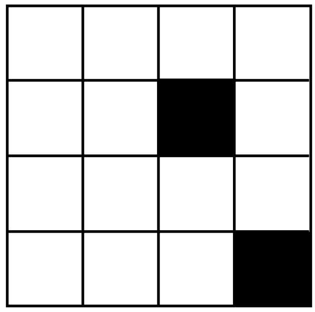
(b) Provide shapes/patterns and ask the students to add to them to match a given statement. For example, "Add to this shape so that it has rotational symmetry of order 3."



"Add to this shape so that it has rotational symmetry of order 2."

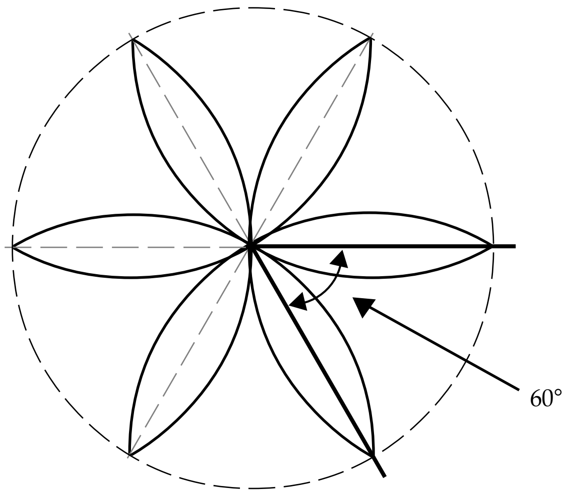


"Complete this grid so that it has rotational symmetry of order 4."

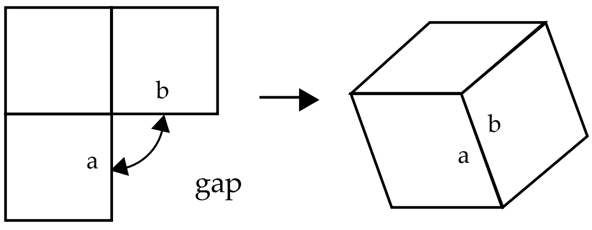


The students should be encouraged to create their own problems, which they can then exchange with other students in their group.

(c) The students explore the link between the order of rotational symmetry and the number of degrees in a circle. This will involve proportional reasoning. To create a figure with rotational symmetry, an element is created then rotated $\frac{1}{6}$ of 360 = 60°.



Use the properties of the interior angles meeting at a point to see what happens if the angles do not total 360°. For example, try putting three squares together around a point. If you do this on a two-dimensional shape, it leaves a gap, but if you introduce a third dimension and make a cube, the sides can meet.



Putting three triangles together around each corner results in a tetrahedron, four triangles around each corner results in an octahedron and five triangles around each corner results in an icosahedron (20 faces). "What would be the recipe for a dodecahedron (a solid with 12 pentagonal faces)?"

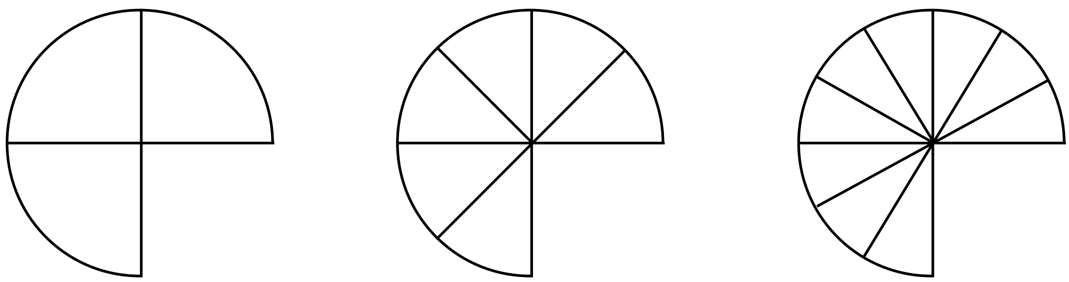
Algebra Links to the Number Framework

There is on-going debate among mathematics educators as to what algebra is and at what point students start learning algebra. It is clear that when students are manipulating expressions and equations that contain letters and operational symbols, such as, $7p^2 - 6 = q$, they are working in the realm of algebra. Of more interest from a teaching perspective is what learning experiences help students to make sense of algebra.

Modern approaches to teaching algebra use either geometric, numeric or measurement contexts to generate relationships. For example, students might look at how the number of pumps affects the circumference of a balloon as it inflates (measurement) or how many square tiles are needed to make each successively larger square (geometry). The central theme of both these approaches is the development of an idea of relative change.

Traditional algebraic tasks, such as evaluating $2x + 5 = 29$, create the perception that letters represent "numbers in disguise". In this case, x is a specific unknown, 12. However, the real power of algebra lies in its ability to elegantly describe generalised relationships between variables and how these variables change relative to one another. For example, $c = \pi d$ describes how the circumference of a circle (c) changes as its diameter (d) changes (π is a fixed number, 3.14...). This relationship applies to *all* circles. It is the *all* feature that makes algebra such a powerful thinking tool. Some letters have standardised use as variables, for example, a for area, b for base, h for height in the formula for the area of a triangle: $a = \frac{1}{2}bh$.

The number strategies that students bring to algebraic instruction affect the students' ability to recognise and describe relationships between variables. For example, consider the situation where students have been provided with fraction circle pieces and asked to make three-quarters of a circle using identical pieces each time. The students produce these models:



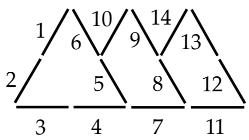
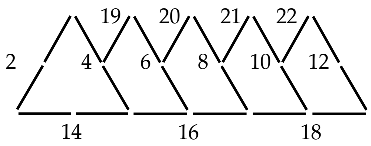
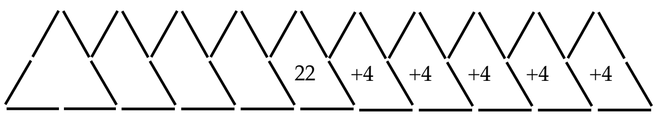
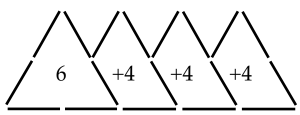
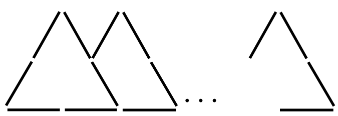
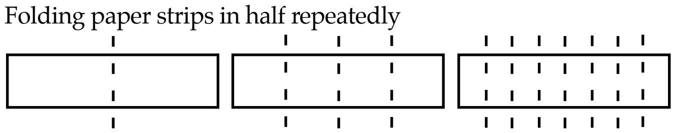
$\frac{3}{4}$ $\frac{6}{8}$ $\frac{9}{12}$

The teacher asks the students to describe what patterns they notice in the symbols for the equivalent fractions of $\frac{3}{4}$. Some students note that the numerators of the fractions are increasing by threes, 3, 6, 9, while the denominators are increasing by fours, 4, 8, 12. These students are applying additive thinking, which, while correct, is insufficient for them to be able to generalise the relationship between the numerator and denominator of *all* fractions equivalent to $\frac{3}{4}$.

To extend the students' thinking, the teacher asks them to write another fraction that is equivalent to $\frac{3}{4}$. The additive thinkers write fractions like $\frac{12}{16}$ and $\frac{15}{20}$ by extending their add three and add four rules for the numerator and denominator. Students who prefer doubling write fractions like $\frac{18}{24}$ and $\frac{36}{48}$. By contrast, the students with multiplicative strategies write fractions like $\frac{75}{100}$ and $\frac{30}{40}$. Their explanations are multiplicative, for example, "If you divide the numerator by 3, then multiply the answer by 4, you get the denominator." These students are providing a generalised statement about all fractions that are equivalent to $\frac{3}{4}$.

This teaching and learning activity is both number and algebra. It is the generalisation that makes the activity algebra, that is, finding a general rule that applies to all fractions that are equivalent to $\frac{3}{4}$.

The table below links strategy stages of the Number Framework with the generalisation of sequences and patterns. This is the teaching approach to algebra advocated in *Mathematics in the New Zealand Curriculum.* Note that attention to spatial or physical attributes can help students learn about the nature of the number relationships. At times, however, it can complicate the algebra teaching point by distracting the learner with too much information.

Number Strategy Stages	**Sequences and Patterns**	**Example**
Count All	Continue a simple pattern of objects and find the number of objects, using one-to-one counting	
Advanced Counting	Continue simple patterns of objects and find the number of objects by systematic counting in groups	
Early Additive	Continue patterns of objects and predict the number of objects in an extension of the pattern, using repeated addition or systematic counting	
Advanced Additive	Anticipate the number of objects in extensions of patterns, using multiplicative strategies for linear patterns and additive strategies for more complex patterns	 10 mountains take 6 + (9 × 4) sticks or (10 × 6) – (9 × 2) sticks
Advanced Multiplicative	Reverse multiplicative rules to anticipate how many elements of a linear pattern can be built with a given number of objects	 You have 102 sticks. 1 mountain uses 6 sticks. 102 – 6 = 96. Each additional mountain uses 4 sticks. 96 ÷ 4 = 24. 24 + 1 = 25, so 25 mountains were made.
Advanced Proportional	Use and manipulate multiplicative rules for linear patterns and recognise patterns in differences and common ratios between members of sequences	Folding paper strips in half repeatedly  Folds 1 2 3 Creases 1 3 7 Sections 2 4 8 Number of sections is 2 to the power of number of folds.

Note that the use of letters to express generalisations is neither recommended nor discouraged. Some research indicates that young students are able to use letters with understanding given appropriate instruction. Teachers should feel free to use letters if they feel it will assist their students to communicate ideas. Algebraic thinking is embedded in part-whole number strategies, and the use of letters to describe generalisations is entirely appropriate for students at the higher stages of the Framework.

The following table provides descriptors of algebraic thinking derived from generalised arithmetic for stages of the Number Framework.

Number Strategy Stages	**Generalised Arithmetic**	**Examples**
Advanced Counting	The student recognises that counting the objects in a set tells how many.	The student knows that the total number of objects is the same no matter which order the objects are counted in and solves addition problems by counting on from the largest number, e.g., $4 + 9$ as 10, 11, 12, 13.
Early Additive	The student recognises that two or more sets can be redistributed without changing the total number of objects.	The student solves addition and subtraction problems by reorganising known facts, e.g., $8 + 7$ as $7 + 7 + 1$ or $9 + 6 = 10 + 5$. The student can write appropriate problems where these strategies could be used.
Advanced Additive	The student states generalisations in words, symbols and/or letters that apply to statements involving addition and subtraction.	The student can find multiple solutions to equations with related unknowns, e.g., $567 + \square = 570 + \bigcirc$ or $\square - 48 = \triangle - 50$, and can make generalised statements, using words or letters about the numbers in the boxes, e.g., 567 + *"number"* = 570 + *"3 less than number"*, or, $q - 48 = (q + 2) - 50$.
Advanced Multiplicative	The student states generalisations in words, symbols and/or letters that apply to statements involving multiplication and division.	The student can find multiple solutions to equations with related unknowns, e.g., $12 \times \square = 4 \times \triangle$, or, $\square \div 20 =$ ⬡ $\div 2 \div 5$, and can make generalised statements, using words or letters, about the numbers in the boxes, e.g., $12 \times j = 4 \times (3 \times j)$, or, $k \div 20 = \frac{1}{2} k \div 2 \div 5$.
Advanced Proportional	The student states generalisations in words, symbols and letters that apply to statements involving fractions, ratios and proportions.	The student can find multiple solutions to equations with related unknowns, for example, $\frac{4}{\square} = \frac{\triangle}{9}$, or, $\square \times 12 = 12 \div \bigcirc$, and can make generalised statements, using words or letters about the numbers in the boxes, e.g., $a \times b = 36$, or, $z = \frac{1}{y}$

Learning Experiences to Develop Students' Reasoning in Algebra

The lessons that follow are examples of algebra teaching aimed at getting students to use generalised thinking. Both geometric and arithmetic patterns are illustrated as vehicles for teaching and learning algebra.

Thinking Ahead

I am learning to copy and extend a repeating pattern and use number strategies to think ahead.

Links to Other Strands

Geometry: naming shapes and describing their features

Number: sequences of ordinal numbers, 1st, 2nd, 3rd, 4th, ...
repeated sequences of numbers, like multiples, for example, 0, 5, 10, 15, 20, and odd numbers, that is, 1, 3, 5, 7, ...

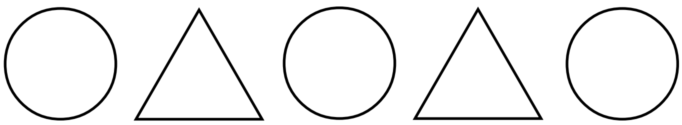
Required Knowledge

- Vocabulary to describe objects, for example, "triangle", "square", "blue", "large"
- One-to-one matching for copying the pattern
- Skip-counting for prediction

Equipment: Attribute blocks, pattern blocks, novelty counters (for example, teddies, fruit, dinosaurs), repeating pattern sheets, Material Master 9–3.

Using Materials

Provide the students with repeating patterns made with the equipment. For example:

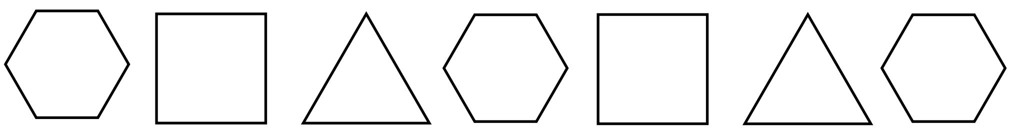


. . .

Ask the students to copy the pattern and, if successful, extend it further. Discuss how they decided which object came "next". Reciting the sequence of shapes can help, for example, "Circle, triangle, circle, triangle, circle, ..."

Vary the difficulty of the patterns by:

(i) increasing the number of objects in the repeating element, from two to three to four, etc. objects. For example:



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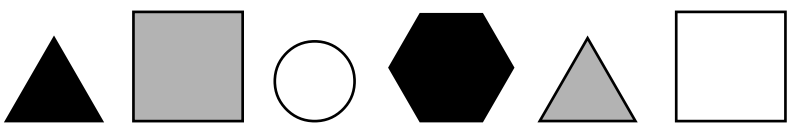
(ii) increasing the number of variables in the pattern, from one to two to three variables, for example, shape and colour; shape, colour and size.



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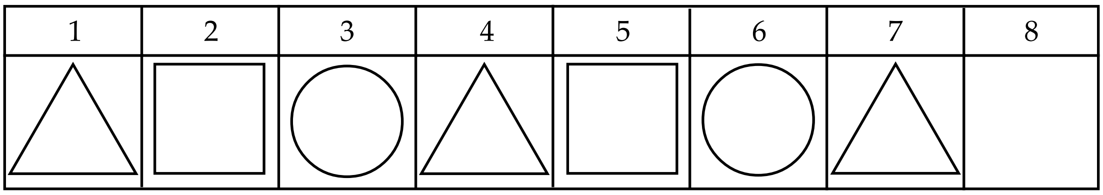
Note that increasing the number of variables makes a marked difference to the difficulty of continuing the pattern. It is also important to ensure that enough members of the sequence are available initially so that the pattern is discernable (for patterns of numbers at least four numbers are needed). Reciting the word sequences can also help, but there will be separate sequences for each variable, for example, "Triangle, square, circle, hexagon, triangle, ...," and, "Black, grey, white, black, ...". Get the students to build patterns for other students to copy and continue.

Using Imaging

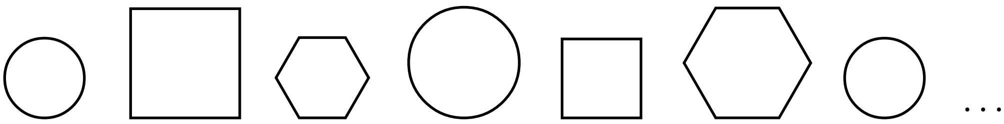
Use the repeating patterns sheets (Material Master 9–3) as a base to build the repeating patterns on. For example:



Ask the students to predict what shape will be in different terms of the sequence, for example, "What will be the fifteenth shape? ... the twenty-third shape? ...". Encourage the Advanced Counting students to use skip-counting strategies to find the shape rather than resorting to building the pattern and using one-to-one counting. For the pattern above, the students may recognise that it repeats in blocks of three shapes. So shapes 3, 6, 9, 12, 15, ... will all be circles. Get the students to create patterns for others to predict. The students could draw these patterns and add them to a collection of pattern puzzles for future use.

Using Algebraic Properties

Provide examples of sequential patterns without the support of the repeating pattern sheets and ask the students to solve predictive problems involving ordinal numbers. For example:



"What will be the seventeenth shape?"

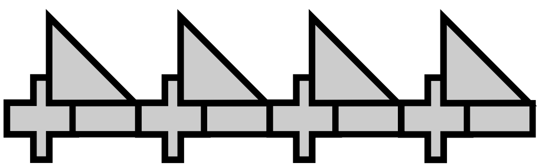
Sticky Moments

I am learning to use my number strategies to predict how many objects will be needed to make a pattern.

Links to Other Strands

Number: multiplication as repeated addition (See *Book 6: Teaching Multiplication and Division*, pages 2–11.)

Geometry: symmetric patterns that involve translation, for example:



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Required Knowledge

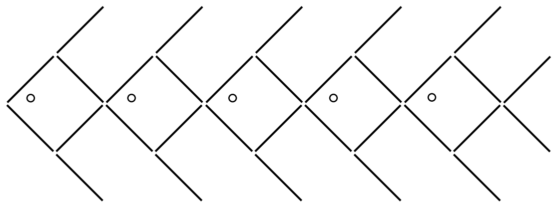
Students need to be able to:

- continue a repeating sequential pattern
- describe shapes using the vocabulary of geometry, for example, "triangle", "rectangle"
- solve arithmetic problems using advanced counting at least.

Equipment: Nursery sticks, iceblock sticks, or matchsticks.

Using Materials

Show the students this geometric pattern of fish made with nursery sticks, iceblock sticks or matchsticks.

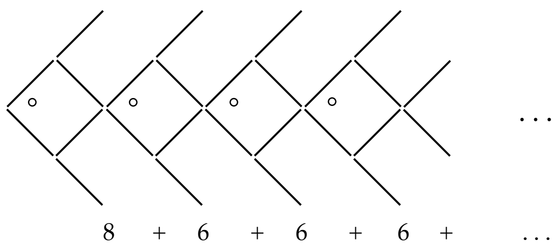


Ask the students to find out how many sticks were needed in total to build the 5 fish. Discuss the strategies they used to solve this problem. Strategies will range from one-to-one counting to using "chunks" of the pattern to simplify the counting, for example, the interior diamonds take 4 sticks each, $5 \times 4 = 20$ sticks, and the fins and tails take $6 \times 2 = 12$ sticks. Ask the students to work out how many sticks would be needed to make 20 fish in this pattern. Provide them with sticks, should they be needed, but do not encourage the students to physically build the pattern.

After an appropriate time, ask the students to share their strategies in small groups. The strategies might include:

(i) **Building and systematic counting**
(Students up to the Advanced Counting stage will tend to use these strategies.)

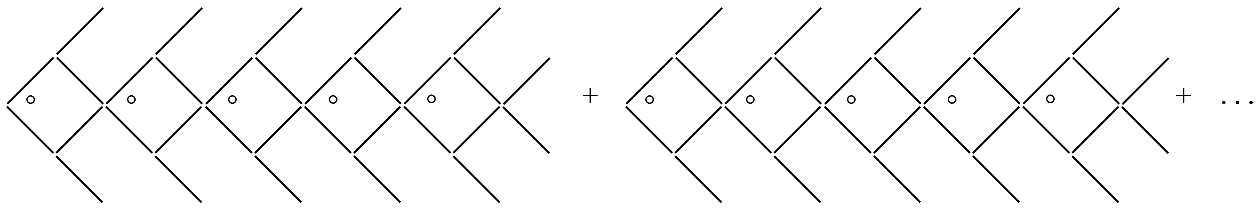
(ii) **Recursive, Additive**
(Students at the Early Additive stage tend to use these strategies.)



(iii) **Ratio**
(This is a signal that the students are using multiplicative reasoning. However, it is an incorrect strategy.)

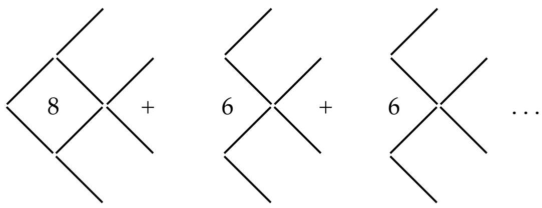
5 fish take 32 sticks, so the reasoning follows that 20 fish take $4 \times 32 = 128$ sticks (that is, $5 : 32 = 20 : 128$).

Note that this method of multiplying 4 lots of 5 fish overlooks the fact that 3 "tails" are effectively lost when the groups are joined nose to tail. Consequently, there are 6 too many sticks counted using this ratio strategy.

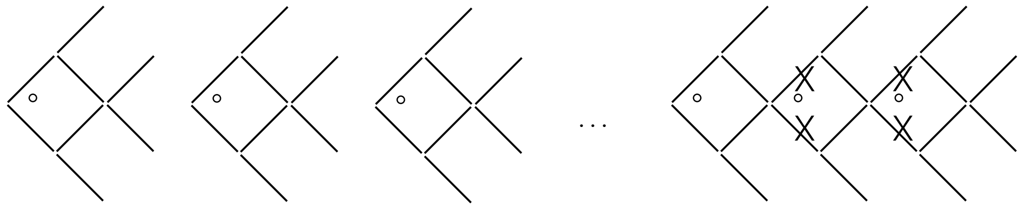


(iv) **Direct, Multiplicative**

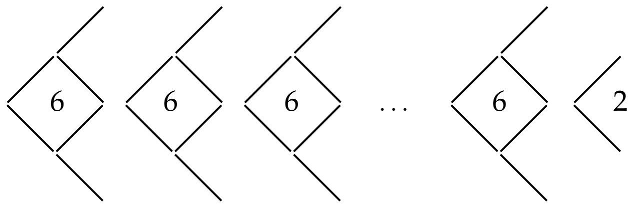
(Students from the Advanced Additive stage onwards use these types of strategies.)



8 + (19 × 6) = 122 (8 sticks for the first fish + 6 sticks each for the other 19 fish)



(20 × 8) – (19 × 2) = 122 (20 lots of 8 sticks for 20 complete fish less 19 lots of two sticks for the sticks that serve as both nose and tail when the fish are joined together)



(20 × 6) + 2 = 122 (6 sticks for each fish body plus 2 sticks for the final tail)

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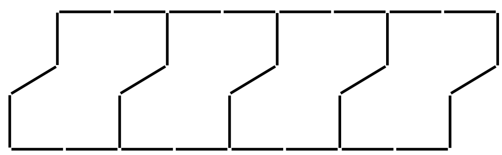
Using Imaging

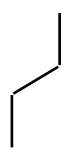
Provide examples of other stick patterns and strategies that have been used by other hypothetical students to find the number of sticks for a given pattern number. For example:



"Jody worked out how many sticks were needed to make 10 seats in this pattern.

She used this strategy; (10 × 4) + (11 × 3). Where did Jody get these numbers from?"

Ask the students to focus spatially on the pattern and to discuss which features of the pattern Jody may have got the lots of 4 and 3 sticks from. The students should notice that in 10 seats there will be 11 groupings of



and 10 groupings of

"For the seats pattern:

Henare wrote, $(10 \times 10) - (9 \times 3)$. Where did he get these numbers from? (10 complete seats less 9 joins)

Zoe wrote, 10 + 7 + 7 + 7 + 7 + 7 + 7 + 7 + 7 + 7. Where did her numbers come from?" (10 sticks for the first seat plus 7 sticks for each new seat)

Using Algebraic Properties

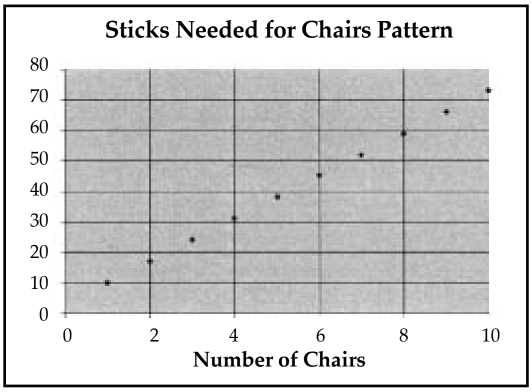
Extend work on the stick patterns by getting the students to apply the rules they have created to the following problems:

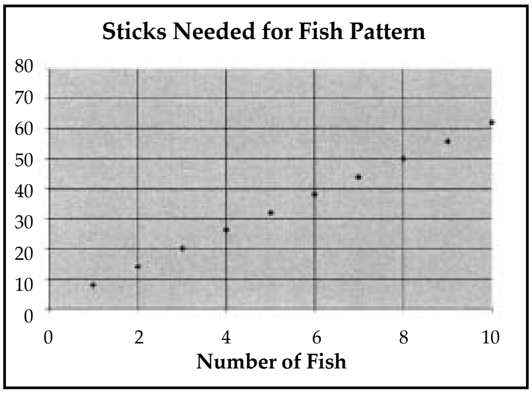
(i) Find the number of sticks to make a given term in a pattern. For example, "How many sticks are needed to build 47 fish?" Note that extrapolating the pattern like this will encourage multiplicative reasoning since repeated addition is very cumbersome.

(ii) Give the students the number of sticks used and ask them to find the number of terms in the pattern. For example, "If 115 sticks were used, how many seats were made?" This involves reversal of the multiplicative rules or trial and improvement strategies and is most appropriate for Advanced Multiplicative students. For the rule, "The number of seats multiplied by 7 add 3 gives the number of sticks used ($s = 7c + 3$; where s = sticks and c = seats)", the students might try using a rounded number of seats, for example, 10 seats need 73 sticks, 20 seats need 143 sticks. From this, they may close in on the correct number of seats by trial and improvement.

Alternatively they might take 3 away from 115 to get 112, then divide 112 by 7 to give 16, the correct number of seats.

(iii) Use a computer spreadsheet to graph the number of seats and the number of sticks and/or extrapolate the relationship further using the "fill down"capability (see Figure It Out, Algebra, levels 3–4, *Answers and Teachers' Notes*, pages 20 for more details on this process). Setting up the graph within the spreadsheet rather than as a separate document allows the students to explore how the graph changes when the starting number or the difference between the terms is altered.





The constant difference between the terms of a pattern determines the slope of the line of points while changes to the starting number raise or lower the line. This leads to the slope and intercept (y value when x is 0) of linear functions.

Related Activities

Figure It Out, *Algebra*, levels 2–3, 3, 3–4 and year 7–8 (four books) contain many examples of spatial patterns that can be used to develop algebraic thinking in this way. The website nzmaths.co.nz also has a collection of algebra units that can be downloaded for classroom use.

Numbers at Work

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I am learning to describe how a number strategy works and to show which numbers it works for.

Links to Other Strands

Mathematical Processes: classification and making conjectures from the Logic and Reasoning sub-strand

Required Knowledge

The students will need to be able to:

- use mathematical vocabulary that classifies sets of numbers, for example, “sum”, “factors” (The students’ proficiency needs to develop as they progress through the strategy stages.)
- model number strategies with appropriate equipment.

Equipment: Place value materials such as interlocking cubes, bundles of iceblock sticks, canisters of beans, place value blocks, etc, Slavonic abacus, cubes, counters, plastic cups, Material Master 9–4.

Using Materials

Begin the lesson by introducing a scenario of a student solving a number problem using a particular strategy. The complexity of the strategy will vary depending on the stages that the students are at. For example:

Advanced Counting
“Leeana has the problem 6 + 17. She starts on 17 and counts on 6 more to get 23.”

Early Additive
“Jacques solves the problem 8 + 29 by moving one from 8 and putting it on 29 to make 7 + 30. He knows this is 37.”

Advanced Additive
“Ani solves the problem 83 – 37 by working out 83 – 40 = 43 then adding 3 to get an answer of 46.”

Advanced Multiplicative
“Callum solves the problem 6×33 by working out $3 \times 33 = 99$ then $2 \times 99 = 198$.”

Advanced Proportional
“Trudi has to find three fractions between $\frac{3}{7}$ and $\frac{4}{7}$. She thinks that $3\frac{1}{4}$, $3\frac{1}{2}$, and $3\frac{3}{4}$ sevenths will work because she knows that $3\frac{1}{4}$, $3\frac{1}{2}$, and $3\frac{3}{4}$ all lie between 3 and 4. She renames the fractions as $\frac{13}{28}$, $\frac{7}{14}$, $\frac{15}{28}$.”

In each case, the students should be asked to demonstrate with appropriate materials how the student in each example is solving the problem. For example, Early Additive students might use place value materials while Advanced Proportional students might use regions like circles or squares.

Using Imaging and Number Properties

Then take the students through the following sequence of problem steps. At each step, it may be necessary to refer back to using materials. Below we use the Advanced Multiplicative stage example to illustrate the steps. In this case, animal strips or place value materials could be used as the equipment to model the equations, should students need it.

Problem Type	**Example**
Initial strategy example	Callum solves the problem 6×33 by working out $3 \times 33 = 99$ then $2 \times 99 = 198$.
Related examples	How would Callum use the same strategy to work out: 4×37? 8×46? 12×17?
Verifying trueness and falseness	Callum knows which of these equations are true without working out the numbers. Which of these equations are true? $4 \times 93 = 2 \times 93 \times 2$ $16 \times 12 = 16 \times 2 \times 3 \times 3$ $5 \times 2 \times 46 \times 3 = 30 \times 46$
One unknown	Use Callum's strategy to work out what number goes in each box: $9 \times 756 = \square \times 3 \times 756$ $\square \times 3.47 = 5 \times 3 \times 3.47$ $7 \times 2 \times \square \times 87 = 87 \times 28$
Two or more unknowns	What numbers could be entered in the shapes to make these equations correct? $\square \times 75 = 75 \times \triangle \times \bigcirc$ $\bigcirc \times \triangle \times$ ⬡ $\times 6.03 = \square \times 6.03$
Variable relationships	In this equation, what is always true about the numbers that go in the different shapes? $\square \times 13 = 13 \times \triangle \times \bigcirc$
Letters to describe variable relationship	Callum has used letters to name the numbers that could go in some boxes. What could he say about the numbers that would go in the empty boxes? $2 \times 13 \times \square = 13 \times q$ $5 \times \square \times 6.4 = h \times 6.4$ $\square \times w \times 91 = u \times 91$

Material Master 9–4 shows how these steps apply to strategies from the other stages.

Related Activities

For ideas about how to develop notation using letters, refer to the nzmaths.co.nz algebra unit called "Cups and Cubes".

Patterns of equations can be used in the same way as strategies to develop ideas about variables. The equation set below is an example of this.

$1 + 2 + 3 = 6$

$2 + 3 + 4 = 9$

$3 + 4 + 5 = 12$

$4 + 5 + 6 = \square$

(i) Write the next equation in this pattern.

(ii) Write an equation in this pattern that is much further down the pattern.

(iii) The equation shown below fits the pattern. The letter 'm' has been used to show one of the numbers. What are the other numbers?

$$\square + m + \bigcirc = \triangle$$

Answer:

$$\boxed{m - 1} + m + \bigcirc\!\!\!\!\!\;(m + 1) = \triangle\!\!\!\;(3m)$$

Statistics Links to the Number Framework

Modern thinking in statistics emphasises the need for students to undertake statistical investigations. There are two main types of investigation. In the first type, students pose questions, gather data and use the data to answer the questions. In the second type, students look for patterns and trends in existing data sets and generate questions to be answered. While traditional teaching of statistics favours the first type of investigation, the modern approach emphasises the second type. In both cases, there is an investigative cycle.

A Question-driven Investigative Cycle

A question-driven investigative cycle follows these stages:

- Formulate a question.
- Gather suitable data.
- Present the data in a sensible way.
- By discussing the trends/patterns in the data representations, answer the question.
- From the investigation, more questions may arise and the cycle recommences.

A Data-driven Investigative Cycle

A data-driven investigative cycle is quite different. Instead of formulating a question first, existing data sets with many variables are located. The students display two variables at a time on graphs. This assists them in formulating questions. Sometimes "eyeballing" the raw data without graphing can also generate questions.

Example: Data has been gathered about used cars for sale and entered into a spreadsheet:

Car Model	Automatic or Manual	Year of Manufacture	Odometer (-000 km)	Power (kw)	Colour	Cost ($)	...
4-door sedan	Automatic	1996	67	88	Red	$13,500	...
Hatchback	Automatic	2001	49	87	Yellow	$28,000	...
Station wagon	Manual	1999	78	68	Red	$21,000	...
2-door sedan	Automatic	1998	152	98	Black	$11,000	...
4-wheel drive	Manual	2003	134	122	White	$34,000	...
							...

A student might then choose to:

- compare year of manufacture against odometer reading
- compare the cost against the car model
- order the data by distance travelled, using the spreadsheet Sort function, and look for patterns in the other variables.

A data-driven investigative cycle follows these stages:

- Find sources of interesting data with lots of variables.
- Eyeball the raw data for patterns and/or use suitable software or other methods to present pairs of variables in the data in a meaningful way.
- Formulate a question.
- Answer the question.
- Recommence the cycle by investigating other relationships in the data.

Statistical Skills

Traditional statistics teaching often focused on developing skills that were isolated from an investigative cycle. For example, students were asked to work out the average from data when finding the average did not supply any information that was needed. This is not to denigrate the need for skills but merely to point out that these skills need to be used in a context of answering pertinent questions. Some such skills include:

- formulating questions
- designing questionnaires
- planning to collect data
- collecting data
- inputting the data into a computer
- using relevant software to display data meaningfully
- using means and medians (when this assists in answering the questions).

Choice of Data Presentation Software

Software for both PCs and Macs that allows data to be presented graphically is normally supplied in a package on the purchase of the computers. This software may include many powerful statistical features that are useful for experienced users. However, other more student-friendly software is becoming available for student use while they are learning how to undertake statistical investigations.

References to such software can be found at:
www.nzmaths.co.nz/statistics/Softwarelist.aspx

Schools should consider the value of such software balanced against any costs involved.

Sources of Questions and Data

Figure It Out

The statistics booklets in the Figure It Out series are:

- *Statistics*, levels 2–3
- *Statistics*, level 3
- *Statistics*, levels 3–4
- *Statistics*, level 4, Book One
- *Statistics*, level 4+, Book Two.

These booklets contain an extensive range of activities and investigations. In some cases, data has already been collected and presented in graphical form ready for interpretation. Some activities and most investigations will need to be adapted for use in a data-driven investigative cycle. A guide for teachers is offered in the *Answers and Teachers' Notes* booklet that accompanies each level of student book. These teacher guides are a valuable source of teaching ideas.

CensusAtSchool

CensusAtSchool NZ is part of a worldwide programme that produces sets of data about school students. It is designed to:

- provide real datasets about students for students
- raise awareness of what a national census is
- use ICT to enhance learning and teaching of data handling.

Students can access the website: www.censusatschool.org.nz/survey.php to find suggested questionnaires and advice on how to conduct question-driven investigations.

Introduction to Probability Ideas

Initially, concepts about probability appear to be simple. For example, the idea that there is a 50% chance that, on tossing a coin, heads will come up is readily accepted and apparently well understood. It is based on the simple idea that there are two equally likely outcomes – a head and a tail. However, there are at least two complicating factors; the past in this case does not affect the future, and, when tossing a coin repeatedly, there is no reason to expect to end up with a head exactly 50% of the time.

Many students and adults believe that what has gone before can affect the future when it can not. For example, in one summer series of cricket test matches, the New Zealand cricket captain lost six tosses in a row. Some people believed that the chance of his winning the next (seventh) toss was better than 50%. This was untrue. Teachers may discuss this with their students, and model it with experiments, but they will not necessarily convince the doubters.

A more profound difficulty is to understand what a 50% chance actually means. It does not mean that 50% of the tosses will result in heads. For example, in 13 tosses there cannot be $6\frac{1}{2}$ heads. What it *does* mean is that, as more and more tosses are made, the proportion of heads varies. The variation in the result gets less and less as the number of tosses increases and the proportion of heads being the result of each toss gets closer and closer to 50%. Probability is unavoidably linked to the idea of long-term proportion (long-run relative frequency).

Variation

Why bother to get into the idea of variation with students? At some point, the idea of probability as long-term proportion (long-run relative frequency) needs to be addressed with students. Then variation can no longer be avoided.

Exact probabilities are seldom available in the real world. They can only be estimated by finding a proportion from a number of trials. For example, in an experiment about tossing a drawing pin and finding the proportion of times the pin lands point up, there is no correct simple theory to predict the probability of "point up". Students often reason there are two possibilities, "point up" and "point down", so the probability of "point up" must be 50%.

Unfortunately this reasoning fails on the grounds that the two outcomes are not equally likely. The only reliable way of finding the probability is by repeatedly tossing the pin and processing the results. Suppose, in an experiment, the pin lands point up 67 times in 100 tosses. The proportion of "point ups" so far is 67%. Does it mean the probability of "point up" is 67%? Unfortunately, no. Suppose after another 100 tosses there is a total of 138 "point ups". Now the estimate is $\frac{138}{200} = 69\%$.

So the estimate of the probability of "point up" changes as the number of trials changes. Therefore, it is important students plot a "moving" proportion on a graph and observe the trend. Students need to see that, as the number of tosses increases, the estimate moves increasingly towards the actual probability of tossing a "point up" but the real probability can never be known exactly.

Learning Experiences to Develop Students' Reasoning in Statistics

Investigative Cycle Activities

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Question-driven Investigation

I am learning to use a question-driven statistical investigation cycle.

Equipment: Computers, website: www.censusatschool.org.nz/survey.php, Figure It Out statistics books (see page 42).

The students will undertake a question-driven investigative cycle as described below:

- Formulate a question.
- Gather suitable data.
- Present the data in a sensible way.
- Solve the answer to the question by discussing the trends/patterns in the data presentations.
- Possibly generate more questions and recommence the cycle.

Data-driven Activities

I am learning to use a data-driven statistical investigation cycle.

Equipment: Computers, website: www.censusatschool.org.nz, Figure It Out.

Sample Investigation

Show the students this spreadsheet data of medal counts from the 1996 Atlanta Olympics (available from www.nzmaths.co.nz/statistics/Investigations/Atlantamedals.htm).

	A	B	C	D	E	F	G
1		26 Top Countries by Gold Medals					
2		at the 1996 Atlanta Olympics					
3							
4	Country	Population	Gold	Silver	Bronze	Total	Medals per
5		(millions)					million
6	United States	265	42	32	25	99	0.37
7	Russia	147	26	20	14	60	0.41
8	Germany	82	18	16	27	61	0.74
9	China	1210	16	22	12	50	0.04
10	France	58	15	7	14	36	0.62
11	Italy	57	12	8	11	31	0.54
12	Australia	18	9	9	20	38	2.11
13	South Korea	45	7	13	5	25	0.56
14	Cuba	11	7	7	8	22	2.00
15	Poland	39	7	5	4	16	0.41
16	Ukraine	52	7	2	11	20	0.38
17	Spain	39	5	6	5	16	0.41
18	Hungary	10	5	4	9	18	1.80
19	Romania	23	4	7	8	19	0.83
20	Greece	10	4	4	0	8	0.80
21	Switzerland	7	4	2	0	6	0.86
22	Denmark	5	4	1	1	6	1.20
23	Turkey	64	4	1	1	6	0.09
24	Canada	29	3	10	8	21	0.72
25	Bulgaria	8	3	6	5	14	1.75
26	Japan	125	3	6	5	14	0.11
27	Netherlands	15	3	5	10	18	1.20
28	Czech Republic	10	3	3	4	10	1.00
29	Brazil	157	3	2	9	14	0.09
30	New Zealand	3.5	3	2	1	6	1.71
31							

Generating Questions

Eyeballing the Raw Data

The students should realise that graphs are the main tool available to generate questions, but not the only one. Often just "eyeballing" the raw data can generate questions. For example, "Consider how the data in the table (above) has been arranged."

"Is that fair?"

"Which country really did best in the total medals considering population?"

"How would you present the data to best show this?"

New Challenge

Have the students add a column in the spreadsheet to enable them to compare gold medals per million people and country and ask them which country is top for gold medals.

Plotting the Data

Plotting graphs showing one variable against another is a powerful way to generate questions. For example, have the students plot gold medals against silver medals and ask them whether there is a relationship. Have the students plot gold medals against bronze medals and ask them whether there is a relationship.

Recycling

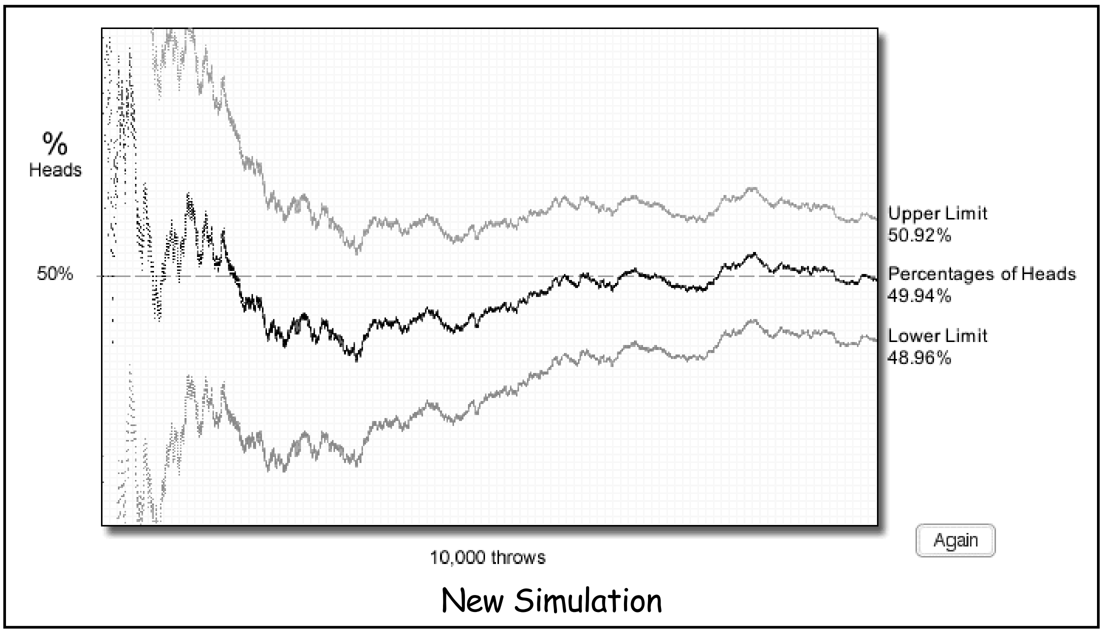
Go back to asking further questions and answering them, revisiting the cycle.

Probability Linked Activities

These activities are designed to help the students understand the idea that variation in proportion (long-term frequency) decreases as the number of experiments increases. The activities are quite subtle and require great care in teaching.

Coin Tossing on the Computer

This lesson uses a computer simulation of 10 000 tosses of a coin. It calculates and graphs the proportion of heads continuously as the number of tosses increases. The centre line shows the proportion of heads. The outer lines show the "margins of error."



I am learning how long-term proportion gets closer and closer to a known probability.

Equipment: Simulation software
(available at: www.nzmaths.co.nz/statistics/Probability/simulation.htm).

Have the students open the simulation. Have the students discuss what the graph shows, then get them to click on the New Simulation button for a new set of 10 000 tosses. They can use this function to do repeated simulations.

Discussion

"Why does the centre line vary up and down?" (Natural variation)

"Why is the pattern different for each simulation?" (Each simulation is unpredictable in how the variation occurs.)

"Why does the centre line always point towards 50% as the number of tosses increases in every simulation?" (In the long run, the proportion approaches 50%, which is the theoretical probability of obtaining a head.)

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The Monty Hall Problem

Background

This is a famous problem based on an American quiz show. On the show, a contestant has won a prize. The contestant is shown three doors. Behind the three doors, there is one car and two goats respectively. The idea is that the contestant picks the door that has the car behind it. The contestant picks one door. Then the host opens one of the other doors; the one that he knows has a goat behind it. He then asks the contestant if they would like to change their choice to the third door or stay with their original choice. The question is, "Should the contestant change their mind?"

This problem has caused great controversy. For more details, read page 78 of *The Curious Incident of the Dog in the Night-time* by Mark Haddon[11]. There are also many websites that have simulations of the problem. Type "Monty Hall" into a search engine to find sources.

Below is a more suitable representation of the problem for the students to work with.

I am learning how to decide an argument by gathering data and presenting it sensibly.

Equipment: Sets of three paper cups and objects that can be hidden under the cups (for example, erasers, cubes).

Have the students play the roles of quizmaster and contestant. The quizmaster places an object under one of three cup while the contestant is not watching. The contestant then points at a cup that the object might lie under. The quizmaster now turns over one of the other cups that they know doesn't cover the object. The contestant decides whether to change their choice. The quizmaster shows which cup is hiding the object.

Play the game two or three times with the whole class involved so that every student understands how to play it.

Have the students work in groups of three. Give each group three cups and an object to hide underneath one of the cups. The group appoints a quizmaster, a contestant and a data recorder. The contestant announces whether they will change their mind or not before the game begins. Once this is announced, the contestant must continue to play this way for 10 games. The recorder carefully notes whether the contestant decided to change their mind or not and records each win and loss on paper. After 10 games, the students rotate roles and play a new series of games. Each time the roles are changed, the new contestant *must* make the opposite choice to the previous contestant. So, if the first contestant changed their mind, then the second does not, and vice versa.

Interpreting the Data – Early Additive Students

Additive students have considerable difficulty with proportions, so the activity will be difficult for early additive thinkers.

Draw the table shown on the next page on the board.

Record the win/loss data, being careful to take account of the contestants' initial choices. The example table below shows the data from Group 1 who found that the contestants who changed their mind won a total of 12 games and lost 8 and the contestant who kept their original choice won 6 games and lost 14.

When all the data has been displayed in the table, get each student to write factual statements based on the information they see there. For example: "When the contestant did not change their mind, they won 22 times and lost 58 times. When the contestant did change their mind, they won 54 times and lost 26 times."

[11] Haddon, Mark (2003). *The Curious Incident of the Dog in the Night-time.* Jonathan Cape: London.

	Changes Mind		Does Not Change Mind	
	Wins	Loses	Wins	Loses
Group 1	12	8	6	14
Group 2				
Group 3				
Group 4				
Group 5				
Group 6				
....	....	....	....	....
Totals	54	26	22	58

Ask the students to discuss what the results indicate, "Should a contestant change their mind or not?" (The evidence will point overwhelmingly to the fact that the chance of a contestant winning is improved substantially by changing their mind. This is despite a very strong feeling among students and many adults that the contestant changing their mind should *not* make a difference.) Encourage the students to write statements about what the experiment shows, such as, "I thought changing my mind would make no difference, but the statistical evidence shows that changing my mind is a good idea."

Interpreting the Data – Advanced Multiplicative Additive Students and Higher

Repeat the data gathering and presentation process as for the stages above. Then use the totals to estimate the probabilities involved as percentages. For example, suppose for six groups of students, the table rows looks like this:

	Changes Mind		Does Not Change Mind	
	Wins	Loses	Wins	Loses
Group 1	6	4	3	7
Group 2	7	3	3	7
Group 3	6	4	4	6
Group 4	8	2	4	6
Group 5	6	4	2	8
Group 6	6	4	2	8
Totals	39	21	18	42

The estimated probability of the contestant winning when they change their mind is $\frac{39}{39+21} = \frac{39}{60} \approx 0.65 = 65\%$

The estimated probability of the contestant losing when they change their mind is $100\% - 65\% = 35\%$

The estimated probability of the contestant winning when they do not change their mind is $\frac{18}{18+42} = \frac{18}{60} \approx 0.3 = 30\%$

The estimated probability of the contestant losing when they do not change their mind is $100\% - 30\% = 70\%$

Ask the students to write statements that involve fractions close to the experimental statistics. For example, "Contestants who change their minds win about two-thirds of the time."

A Final Comment

Some students will enjoy arguing why it is a good idea (or not!) to change one's mind on a purely logical basis. They may like to research the explanations offered on the web. The *correct* probabilities are definitely these:

The probability of winning when a contestant changes their mind is $66\frac{2}{3}\%$

The probability of losing when a contestant changes their mind is $33\frac{1}{3}\%$

The probability of winning when a contestant does not change their mind is $33\frac{1}{3}\%$

The probability of losing when a contestant does not change their mind is $66\frac{2}{3}\%$

Some people find it easier to understand the reasoning for this solution if the problem is extrapolated to 100 items to choose from with only one correct option. In such a case, the contestant would have a 1% (1 out of 100) probability of making the correct choice. Once the quizmaster has removed 98 known "wrong" options, the remaining other option must have a 99% probability of being the winning option – 99% versus 1%. Most people would agree; the contestant should change their mind. So it is for the one in three choice. The contestant has a one-third probability of making the correct choice; there is a two-thirds probability that the quizmaster will have left the correct choice; the contestant should run with the two-thirds probability option and change their choice.

Varying As You Go

In this activity, the students know that the number of red coloured cubes is fixed even though they cannot see them. The task is to find out how many red cubes there are by repeatedly drawing cubes from a box and using statistical reasoning. The students return the cubes to the box after each draw.

I am learning how the data shows a varying proportion as it is collected even though the true proportion is always present.

Equipment: Six cubes of one colour (say red) and five similar cubes of another colour (say blue), a box to shake cubes in, calculators, Material Master 9–5.

Hand out photocopies of Material Master 9–5. Put six red and five blue cubes in a box. Tell the students there are 11 cubes in the box and that they are either red or blue. The students must work out the number of red cubes in the box even though they cannot see into the box. Draw out one cube from the box, then return it. Do this 100 times. After a set of five cubes has been pulled out and then returned, get the students to predict the number of red cubes in the box. Ask the students to fill in a table (as shown in the example below) and graph the results of every five cubes pulled from the box. The students may need help to do the percentage calculations on the calculator.

After 100 cubes have been taken out, ask the students to decide how many red cubes there are. The students will be interested to find out the actual number. Many will be surprised how their predictions can be so inaccurate after such a large number of trials.

Reds (Tally)	Draws (Tally)	Total Reds So Far	Total Draws So Far	Percentage Reds So Far	Comment about the Number of Reds So Far
|||	卌	3	5	3 ÷ 5 = 60%	Some red, some blue
|||	卌	6	10	6 ÷ 10 = 60%	Over a half red
|	卌	7	15	7 ÷ 15 = 47%	Less than a half red
||||	卌	11	20	11 ÷ 20 = 55%	About 6 or 7 red
||	卌	13	25	13 ÷ 25 = 52%	About 6 red
|	卌	14	30	14 ÷ 30 = 47%	Maybe 5 red

The key idea in this lesson is that sampling yields only approximate predictions of the true proportion. The only way to be certain is to actually count all the cubes. Larger samples provide more accurate predictions but they can also be misleading.

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The Two Coin Problem

When probability theory was being developed in the eighteenth century, even famous mathematicians debated what was correct reasoning. This shows that the ideas in mathematical probability are not simple. A dispute broke out between mathematicians about the probability of getting two heads when tossing two coins together. One group of mathematicians argued that there are three ways of the two coins landing: two heads, two tails, and a mix of a head and a tail, so the probability of two heads landing is $\frac{1}{3}$. Another group of mathematicians argued that there are four ways of the coins landing: head and head, tail and tail, tail and head, and head and tail, making he probability of two heads landing $\frac{1}{4}$.

How could a dispute like this be resolved in the classroom? If famous mathematicians disagree, students are also likely to do so, no matter what logical arguments are presented.

I am learning how to decide an argument by gathering data and presenting it sensibly.

Equipment: Coins, Material Master 9–6.

Ask, "Is the probability of getting two heads when you toss two coins together $\frac{1}{3}$ or $\frac{1}{4}$?" Get the students to base their decisions on statistical investigation not logical argument.

Firstly, it is important that the students have the opportunity to see that the estimated proportion varies as data is continually added and that this is a natural feature of variation. The graph is a vital tool for this. Secondly, the amount of variation reduces as the sample size increases. Thirdly, this variation will never be exactly zero even though the variation is reducing as the sample size increases. These are not easy ideas to understand, and they will need to be explained carefully to the students.

Data Gathering

Group the students into pairs. Give each student a photocopy of Material Master 9–6 sheet 1. One student tosses two coins simultaneously. They do this 20 times. The other student makes tally marks at the top of their sheet:

Two Heads	Number of Pairs of Tosses
𝍸 (5)	𝍸 𝍸 𝍸 𝍸 (20)

Record the class data on the board. Get the students to add the class data to their sheets, calculate the percentage, using a calculator, and graph this percentage. For example:

	Two Heads	Number of Pairs of Tosses	Two Heads Running Total	Grand Running Total	Two Heads Percentage So Far
Pair 1	5	20	5	20	5 ÷ 20 = 25%
Pair 2	7	20	12	40	12 ÷ 40 = 30%
Pair 3	2	20	14	60	14 ÷ 60 = 23.3%
Pair 4	5	20	19	80	19 ÷ 80 = 24%
Pair 5	7	20	26	100	26 ÷ 100 = 26%

The students may need help to convert fractions to percentages using a calculator, for example, 5 out of 20 as a percentage is worked out on the calculator as 5 ÷ 20 = 0.2 = 20%.

Have the students add lines to the graph to show $\frac{1}{3}$($33\frac{1}{3}$%) and $\frac{1}{4}$(25%). Discuss as a class what trend the students can see taking place in the graph. It is quite likely the trend will not be clear. The students need to discuss what is the best way to find out the correct probability of getting two heads when you toss two coins together. (An increase in the number of tosses is the only realistic thing that can be done.)

The Drawing Pin Problem

This activity is the first one where the probability of an event *cannot* be found directly by reasoning. The only way to estimate this probability is to undertake an experiment. The students need to plot the estimates of the probability, which means the proportion is continually calculated and graphed. Again the students need to understand that the graph points at the probability without expecting that it will ever arrive at this number. Those who are not at an advanced multiplicative stage may need help with calculating percentages.

I am learning how an estimate of probability improves over time as the number of trials increases.

Equipment: Drawing pins, calculators, Material Master 9–7.

Group the students in pairs. Give each student a photocopy of Material Master 9–7 sheet 1. One student tosses a pin 20 times. The other student makes tally marks at the top of their sheet to show if the pin landed with the point up:

Point Up	Total Number of Tosses
𝍸 𝍸 (10)	𝍸 𝍸 𝍸 𝍸 (20)

Record the class data on the board. For example:

	Point Up	Total Number of Tosses	Point Up Running Total	Running Total of Tosses	Point Up Percentage So Far
Pair 1	10	20	10	20	10 ÷ 20 = 50%
Pair 2	16	20	26	40	26 ÷ 40 = 65%
Pair 3	12	20	38	60	38 ÷ 60 = 63.3%
Pair 4	15	20	53	80	53 ÷ 80 = 66.25%
Pair 5	11	20	64	100	64 ÷ 100 = 64%

Have the students copy the data onto their sheets and fill in the percentage column, then get them to transfer the data onto a graph (scatterplot) that shows running total versus percentage of "point ups".

As a class, discuss the trend shown in the graph. It is quite likely that the trend will not be clear. The students need to discuss what is the best way to find out the correct answer. (Again, increasing the amount of data is the only realistic way of proving the trend.).

ACKNOWLEDGMENTS

The Ministry of Education wishes to acknowledge the following people and organisations for their contribution towards the development of this draft handbook.

THE PARTICIPANTS:

The New Zealand numeracy project personnel – facilitators and principals, teachers, and students from more than eighteen hundred New Zealand schools who contributed to this handbook through their participation in the numeracy projects from 2000–2005.

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Professor Derek Holton, convenor (The University of Otago), Professor Megan Clark (Victoria University of Wellington), Dr Joanna Higgins (Victoria University of Wellington College of Education), Dr Gill Thomas (Maths Technology Limited), Associate Professor Jenny Young-Loveridge (The University of Waikato), Associate Professor Glenda Anthony (Massey University), Tony Trinick (The University of Auckland Faculty of Education), Garry Nathan (The University of Auckland), Paul Vincent (Education Review Office), Dr Joanna Wood (New Zealand Association of Mathematics Teachers), Peter Hughes (The University of Auckland Faculty of Education), Vince Wright (The University of Waikato School Support Services), Geoff Woolford (Parallel Services), Kevin Hannah (Christchurch College of Education), Chris Haines (School Trustees' Association), Linda Woon (NZPF), Jo Jenks (Victoria University of Wellington College of Education, Early Childhood Division), Bill Noble (New Zealand Association of Intermediate and Middle Schools), Diane Leggatt of Karori Normal School (NZEI Te Riu Roa), Sului Mamea (Pacific Island Advisory Group, Palmerston North), Dr Sally Peters (The University of Waikato School of Education), Pauline McNeill of Columba College (PPTA), Dr Ian Christensen (He Kupenga Hao i te Reo), Liz Ely (Education Review Office), Ro Parsons (Ministry of Education), Malcolm Hyland (Ministry of Education).

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The cover design is by Dave Maunder (Learning Media Limited) and Base Two Design Ltd.