MINISTRY OF EDUCATION
Te Tāhuhu o te Mātauranga

Book 8
Teaching Number Sense and Algebraic Thinking

Numeracy Professional Development Projects

EFFECTIVE TEACHING OF NUMBER SENSE AND ALGEBRAIC THINKING

Algebra has always been a difficult area for teachers to teach and for students to learn. There is ongoing debate about when algebra should be introduced and which teaching approaches are most effective. Historically, in New Zealand, algebra has formed an important part of the secondary mathematics curriculum. However, since 1992, teachers have been expected to teach algebra across all levels. This has posed significant challenges for primary teachers.

There is an emerging consensus internationally that the foundations for symbolic algebra lie in students' understanding of arithmetic. So at what point does a student change from operating arithmetically to working with algebra?

One view is that the generalisation of arithmetic operations is algebraic in nature. For example, recognition that "ninety-nine groups of three (99 x 3)" has the same total as "three groups of ninety-nine (3 x 99)" involves algebraic thinking irrespective of whether or not letters are used. Another view is that algebra begins when letters are used to represent these generalisations, for example, $a \times b = b \times a$ (or $ab = ba$). Whatever the view about when algebra begins, making sense of arithmetic is undeniably critical to students' development in algebra and the other areas of mathematics and statistics.

The Numeracy Professional Development Projects place a strong emphasis on students gaining understanding of the number system. This is aligned with trends in modern mathematics education internationally. At all levels of schooling, teachers should encourage students to explore, describe, and generalise structures and relationships through a range of mathematical activities.

Many of these activities will involve the application of number to other aspects of mathematics and statistics. For example, students can investigate the relationships between the perimeter and area of rectangles in geometry or how to convert between related units of the metric system in measurement.

Research about quality teaching[1] shows that students learn most quickly when they have opportunities to "identify and resolve discrepancies between their current understandings and new information". The careful selection of related problems or investigations and the creation of a comfortable classroom climate in which all students can share their mathematical ideas are fundamental to improving achievement. The promotion of creative and efficient recording strategies can also greatly assist students in developing, generalising, and communicating their ideas.

This book provides many examples of lessons in which generalised arithmetic is the vehicle for algebraic thinking. It includes very limited use of algebraic symbols at the upper stages of the Number Framework. Such symbolism is not an end in itself. Often the generalisations expressed by students in words can be represented easily using letters, expressions, and equations. Teachers are encouraged to introduce algebraic symbols when they feel comfortable in doing so and where they feel that their students are likely to benefit.

[1] Alton-Lee, A. (2003). *Quality Teaching for Diverse Students in Schooling: Best Evidence Synthesis.* Wellington: Ministry of Education.

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This book is also available on the New Zealand Maths website, at www.nzmaths.co.nz/Numeracy/2008numPDFs/pdfs.aspx

Teaching Number Sense and Algebraic Thinking

Teaching for Number Strategies

The activities in this book are specifically designed to develop students' mental strategies. They are targeted to meet the learning needs of students at particular strategy stages. All the activities provide examples of how to use the teaching model from *Book 3: Getting Started*. The model develops students' strategies between and through the phases of *Using Materials*, *Using Imaging*, and *Using Number properties.*

Each activity is based on a specific learning outcome. The outcome is described in the "I am learning to ..." statement in the box at the beginning of the activity. These learning outcomes link to the Strategy Learning Outcomes provided in *Book 3: Getting Started.*

The following key is used in each of the teaching numeracy books. Shading indicates which stage or stages the given activity is most appropriate for. Note that CA (Counting All), refers to all three *Counting from One* stages.

Strategy Stage

	Strategy Stage
E →	Emergent
CA →	Counting All (One-to-one Counting, Counting from One on Materials or by Imaging)
AC →	Advanced Counting
EA →	Early Additive Part-Whole
AA →	Advanced Additive–Early Multiplicative Part-Whole
AM →	Advanced Multiplicative–Early Proportional Part-Whole
AP →	Advanced Proportional Part-Whole

The table of contents below details the main sections in this book. These sections reflect the strategy stages as described on pages 15–17 of *Book One: The Number Framework.* The *Key Ideas* section provides important background information for teachers in regard to the development of students' thinking in algebra and number sense.

Strategy Stage(s)	Number Sense and Algebraic Thinking
Advanced Counting	Page 3
Early Additive	Pages 4–5
Advanced Additive–Early Multiplicative	Pages 6–9
Advanced Multiplicative–Early Proportional	Pages 10–36
Advanced Proportional	Pages 37–48

What is Number Sense?

Students with good *number sense* have the ability to judge the reasonableness of numerical outcomes. For example, a student buys a CD player costing $141. She receives a discount of 9.9%. Her number sense enables her to work out the discount approximately:

$9.9\% \approx 10\% = \frac{1}{10}$, $\$141 \approx \$140$. $\frac{1}{10}$ of $140 = $14. So she expects to pay about $141 − $14 = $127. The shop assistant uses a calculator and asks for $133.98. (The shop assistant had wrongly inputted $\frac{9.9}{141} \times 100$ into a calculator and got $7.02.) The student naturally asks for a recalculation. In such a situation, we can say the student has good number sense and the shop assistant does not.

More generally, number sense may be seen this way:

> "Number sense exhibits itself in various ways as the learner engages in mathematical thinking, including awareness of various levels of accuracy and sensitivity for the reasonableness of calculations. It is characterised by a desire to make sense of numerical situations by looking for links between new information and previously acquired knowledge, and an innate drive within the learner to make the forming of these connections a priority."

(From McIntosh, A., Reys, B., Reys, R., Bana, J. and Farrell, B. (1997). *Number Sense in School Mathematics. Student Performance in Four Countries.* Perth: MASTE.)

Understanding Number Properties

In many activities in this book there is a section called **Understanding Number Properties**. These are assessment items designed to help you know whether the students have understood the main ideas in the activity. The items have two different styles. Sometimes the assessment involves using letters. When this proves too difficult for technical reasons, the style of assessment changes and the students are asked to make up an example similar to the ones they have been doing and then solve it.

Activities from Other Books

All activities from *Book 5: Teaching Addition, Subtraction, and Place Value*, *Book 6: Teaching Multiplication and Division*, and *Book 7: Teaching Fractions, Decimals, and Percentages* are referenced for your convenience. The learning objective for each activity is included so you can judge whether that activity is suitable for your students. If it is, refer to the relevant book.

Learning Experiences for Advanced Counting

Activities from Other Books

Number Tiles
Page 18, Book 5.
I am learning to add by counting on when the larger number is given first.

The Number Strip
Page 19, Book 5.
I am learning to add by counting on when the larger number is given first.

The Bears' Picnic
Page 20, Book 5.
I am learning to add by counting on when the larger number is given first.

Frog Jumps
Page 20, Book 5.
I am learning to add by counting on with the larger number first.

The Bigger Number First
Page 21, Book 5.
I am learning to count on from the larger number even when the smaller number is given first.

Change Unknown
Page 21, Book 5.
I am learning to count on to solve problems like $4 + \square = 7$.

Counting Back
Page 22, Book 5.
I am learning to count down to solve subtraction problems.

Adding Tens
Page 22, Book 5.
I am learning to add tens to a number by counting on in tens or adding the tens together.

Subtracting Tens
Page 23, Book 5.
I am learning to subtract tens from a number by counting back in tens or subtracting the tens first.

Adding Ones and Tens
Page 24, Book 5.
I am learning to add two-digit numbers where the sum of the ones digits is less than 10.

Subtracting Ones and Tens
Page 24, Book 5.
I am learning to subtract two-digit numbers that do not involve renaming.

The Missing Ones and Tens
Page 25, Book 5.
I am learning to solve problems like $45 + \square = 67$ which do not involve renaming.

The Thousands Book
Page 25, Book 5.
I am learning to add and subtract ones and tens by moving left, right, up, and down on charts with numbers up to 1 000.

Make Ten
Page 26, Book 5.
I am learning to add three or more numbers by first making up pairs that add up to 10.

Compatible Numbers
Page 26, Book 5.
I am learning to subtract to solve problems like $5 + 3 + 6 - 8$ by first adding five and three to get eight then removing the eight.

Subtraction in Parts
Page 27, Book 5.
I am learning to subtract by splitting numbers into parts instead of counting down.

Up over the Tens
Page 28, Book 5.
I am learning to add by splitting the numbers into parts to make tens.

Number Strips
Page 8, Book 6.
I am learning to solve multiplication problems by counting sets of objects in groups.

Fair Shares
Page 2, Book 7. (2007 edition)
I am learning to find halves, quarters, and other fractions of sets and shapes.

New Activities

There are no new activities for Advanced Counters.

EA
AA
AM
AP

Learning Experiences for Early Additive Part-Whole

Activities from Other Books

Comparisons
Page 29, Book 5.
I am learning to add and subtract using comparisons of sets without counting down or counting up.

More Comparisons
Page 30, Book 5.
I am learning to add and subtract using comparisons of sets without counting down or counting up.

How Many Ten-Dollar Notes?
Page 31, Book 5.
I am learning how many tens there are in numbers less than 1 000.

Animal Arrays
Page 15, Book 6.
I am learning to solve multiplication problems using arrays.

Pirate Crews
Page 17, Book 6.
I am learning to solve problems about sharing into equal sets.

Biscuit Boxes
Page 19, Book 6.
I am learning to solve division problems about how many sets can be made.

Two, Fives, and Tens
Page 21, Book 6.
I am learning to work out multiplication facts from what I know about twos, fives, and tens.

Wafers
Page 6, Book 7. (2007 edition)
I am learning to find fractions of lengths, including seeing when a fraction is greater than one.

Animals
Page 7, Book 7.
I am learning to find fractions of a set.

Fraction Circles
Page 9, Book 7.
I am learning to put fractions in order from smallest to largest.

Hungry Birds
Page 11, Book 7.
I am learning to use addition and subtraction to work out fractions of a set.

New Activities

You Don't Need the Number

I am learning that subtracting then adding the same number leaves the original number unchanged.

Equipment: Tens and ones place-value material.

Using Materials

Problem: "Jean has 45 lollies. She eats 4 of them. Now her grandmother gives her 4 more lollies. How many lollies does Jean have now?"

Write 45 – 4 + 4 on the board. Get the students to model the story with counters. Write the answer on the board.

Examples: Word stories and recording for: 43 + 5 – 5 26 – 6 + 6 52 – 3 + 3 14 + 8 – 14 34 + 3 – 34 ...

Using Imaging

Problem: "Gavin has 56 apples. He eats 6 of them. He buys 6 more apples. How many apples does Gavin have now?"

Shield 56 objects from the students. Show them the 6 Gavin eats. "How many are left?" Add 6 more to the shielded apples. Ask how many there are now.

Examples: Word stories and recording for: 25 + 4 – 4 58 – 7 + 7 28 + 15 – 28 23 – 9 + 9 ...

Using Number Properties

Examples: Word stories for: 345 + 907 – 907 234 + 69 – 234 689 – 341 + 341
345 + 89 – 345 ...

Understanding Number Properties: Find the answer to 500 – $\square$ + $\square$, where the number is the same in both boxes. Would it matter what number went in the boxes? ***(Answer: No. Whatever number is in the boxes, the answer is always 500.)***

Fractions in a Whole

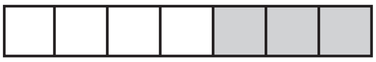
Doing operations with fractions helps the students understand fractions. Operations provide a purpose for using fractions and highlight key structures in the numbers.

I am learning about fractions that add up to a whole.

Equipment: Unilink cubes.

Using Materials

Problem: "Owen makes up a packet of sweets consisting of 4 yellow sweets and 3 blue sweets. What fraction of the packet is yellow?" "blue?"



Discuss why the fractions are $\frac{4}{7}$ and $\frac{3}{7}$. Write $\frac{4}{7} + \frac{3}{7}$ on the board and discuss the answer. *(Answer: $\frac{7}{7}$ or 1 whole.)*

Examples: Repeat using cubes to work out: $\frac{4}{6} + \frac{2}{6}$ $\frac{7}{10} + \frac{3}{10}$ $\frac{8}{11} + \frac{3}{11}$

Using Imaging

Problem: Show the students 5 yellow cubes and hide 2 blue cubes.

"The yellow cubes represent $\frac{5}{7}$ of a packet of sweets. How many blue cubes am I hiding and what fraction of a packet am I hiding?"*(Answer: 2 sweets and $\frac{2}{7}$ of a packet.)*

Examples: Repeat to work out: $\frac{6}{11} + \square = 1$ $\frac{1}{5} + \square = 1$ $\square + \frac{6}{11} = 1$ ...

Using Number Properties

Problem: "Maria bought a box of envelopes for her business. After a month, she finds there is only $\frac{5}{31}$ of the box left. What fraction of the box has been used?"

(Answer: $\frac{31}{31} = 1$ box, so $\frac{31}{31} - \frac{5}{31} = \frac{31-5}{31} = \frac{26}{31}$ must have been used.)

Examples: Work out: $\frac{6}{17} + \square = 1$ $\frac{27}{47} + \square = 1$ $\square + \frac{57}{100} = 1$ $1 - \frac{12}{23} = \frac{\square}{23}$ $1 - \frac{90}{101} = \frac{\square}{101}$ ...

Understanding Number Properties: Write down two fractions whose denominators are both 1 000 and whose numerators make the fractions add up to 1 whole.

Learning Experiences for Advanced Additive–Early Multiplicative Part-Whole

Activities from Other Books

Saving Hundreds
Page 32, Book 5.
I am learning how knowing 10 ones make one 10 and 10 tens make 100 can help me solve problems like $567 + \square = 800$.

Jumping the Number Line
Page 33, Book 5.
I am learning to jump through a tidy number on a number line to solve problems like $17 + \square = 91$.

Don't Subtract – Add!
Page 34, Book 5.
I am learning that problems like $34 + \square = 51$ and $51 - 34 = \square$ have the same answer.

Problems like 23 + $\square$ = 71
Page 35, Book 5.
I am learning to solve problems like $13 + \square = 91$ by jumping up by a tidy number on a number line, then jumping back a small number.

How Many Tens and Hundreds?
Page 35, Book 5.
I am learning how many hundreds there are in numbers over 1 000.

Problems like 37 + $\square$ = 79
Page 36, Book 5.
I am learning to add mentally the ones and tens separately when appropriate.

Problems like $\square$ + 29 = 81
Page 37, Book 5.
I am learning to reverse problems like $\square + 29 = 81$ to $29 + \square = 81$ and then use an appropriate mental method to solve the problem.

When One Number Is Near a Hundred
Page 37, Book 5.
I am learning to solve some addition and subtraction problems by adjusting one number to the nearest hundred.

Problems like 73 – 19 = $\square$
Page 38, Book 5.
I am learning to solve problems like 73 – 19 by first subtracting a tidy number then adding on a small number to get the answer.

Equal Additions
Page 38, Book 5.
I am learning to solve subtraction problems by equal additions that turn one of the numbers into a tidy number.

People's Ages
Page 39, Book 5.
I am learning to apply mental subtraction methods to an application.

A Balancing Act
Page 40, Book 5.
I am learning that the answer on the left of the equals sign is the same as the answer on the right of the equals sign.

Near Doubles
Page 41, Book 5.
I am learning to solve addition problems where the two numbers are easily related to doubles.

Three or More at a Time
Page 42, Book 5.
I am learning to look at the addition and subtraction of three or more numbers to calculate easy combinations first.

Problems like 67 – $\square$ = 34
Page 42, Book 5.
I am learning to solve problems like $67 - \square = 39$ by solving $39 + \square = 67$ or by finding 67 – 39.

Large Numbers Roll Over
Page 43, Book 5.
I am learning how the number rolls over when 10 of any unit occur in an addition, and how the number rolls back when 10 of any unit occur in a subtraction.

Mixing the Methods – Mental Exercises for the Day
Page 43, Book 5.
I am learning to select wisely from my range of mental strategies to solve addition and subtraction problems and discuss my methods with other students.

Mental or Written?
Page 44, Book 5.
I am learning to select wisely between using a mental method and a written method for addition and subtraction problems.

Estimation as a Check
Page 44, Book 5.
I am learning to check any addition and subtraction problem I cannot solve mentally by estimating the answer.

Fun with Fives
Page 28, Book 6.
I am learning to work out my times six, seven, and eight tables from my five times table.

Multiplying Tens
Page 30, Book 6.
I am learning to multiply tens, hundreds, thousands, and other tens numbers.

A Little Bit More/A Little Bit Less
Page 32, Book 6.
I am learning to solve multiplication problems by taking some off or putting some on (compensation).

Turn Abouts
Page 34, Book 6.
I am learning to change the order of the numbers to make multiplication easier.

Long Jumps
Page 36, Book 6.
I am learning to solve division problems using the repeated addition and multiplication facts I know.

Goesintas
Page 38, Book 6.
I am learning to solve "How many sets of …" and "sharing" division problems using multiplication.

Birthday Cakes
Page 14, Book 7. (2007 edition)
I am learning to use multiplication to find a fraction of a set.

Fractional Blocks
Page 15, Book 7.
I am learning to use patterns to find fractions of shapes and sets.

Seed Packets
Page 17, Book 7.
I am learning to find equivalent ratios.

Trains
Page 19, Book 7.
I am learning to find where fractions live amongst whole numbers.

New Activities

Reversing Addition

Problems like 67 + □ = 101 can be solved in a variety of ways. However, when the numbers have many digits, the choice of method effectively reduces to reversing.

> I am learning that problems like □ + 145 679 = 623 455 are best solved by reversing them.

Equipment: Strips of paper. Scissors. Calculators.

Using Materials

Problem: "Murray has $1,858 in the bank. His grandfather put some more money in Murray's account. Now Murray has $5,683. How much money did Murray's grandfather put in Murray's account?"

Write 1 858 + □ = 5 683 on the board. Get the students to make two strips of paper of equal length and label them as shown:

5 683	
1 858	

Discuss why 1 858 off the second strip shows the answer is 5 683 – 1 858 and work out the answer with a calculator.

Problem: "Geraldine collects stamps. Her parents give her a packet of 355 stamps for her birthday. Altogether she now has 6 040 stamps. How many stamps did Geraldine have before her birthday?"

Write □ + 355 = 6 040 on the board. Repeat the method with strips shown above.

Examples: Word stories and recording for: 3 333 + □ = 4 141   □ + 5 601 = 45 893
□ + 7 928 = 30 281 $234.56 + □ = $789.40 ...

Using Imaging

Problem: "Miles has $345, and he wants to buy a mountain bike costing $601. How much more money does he have to save?"

Write 345 + □ = 601 on the board. Ask the students to imagine the strips and solve the problem. Drop back to drawing the strips on the board if needed.

Examples: Word stories and recording for: 4 567 + □ = 6 012   □ + 9 567 = 12 211
□ + 6 443 = 14 601 $291.36 + □ = $1,089.43 ...

Using Number Properties

Examples: Use a calculator to work out these answers: 44 675.83 + $\square$ = 76 645.93

$\square$ + 345.67 = 5 678.09 $444\frac{1}{5} + \square = 800\frac{1}{5}$ 484 609 + $\square$ = 1 678 980

$\square$ + 34.78902 = 56.00912 ...

Understanding Number Properties: $a + \square = b$ where a and b stand for numbers. How would you work out the number that goes in the box? ***(Answer: Work out the answer to b – a.)***

Subtraction to Subtraction

Unlike the examples in the previous activity, 74 567 – $\square$ = 28 973 does *not* turn into an addition. Instead the subtraction is converted into a different subtraction.

I am learning to solve problems like 74 567 – $\square$ = 28 973.

Equipment: Strips of paper. Calculators.

Using Materials

Problem: "Joyce borrows $13,800 to buy a new car. At the end of the year, she still owes $7,890. How much of the loan has she paid?"

Write 13 800 – $\square$ = 7 890 on the board. Get the students to make the following strips:

<table>
<tr><td colspan="2">13 800</td></tr>
<tr><td>7 890</td><td></td></tr>
</table>

Discuss why the answer is 13 800 – 7 890 and then get the students to work out the answer on calculators. *(Answer: 5 910.)*

Examples: Word stories and recording for: 34 567 – $\square$ = 5 678 12.45 – $\square$ = 7.01

20 008 – $\square$ = 13 856 ...

Using Imaging

Problem: "Caterers have 3 190 pies to sell at a rugby league game. At the end of the day, they have 345 pies left. How many pies were sold?"

Write 3 190 – $\square$ = 345 on the board. Encourage the students to imagine the strips to solve the problem. *(Answer: 3 190 – 345 = 2 845 pies.)*

Examples: Word stories and recording for: 4 567 – $\square$ = 2 888 102.45 – $\square$ = 41.89

12 789 – $\square$ = 5 692 23.456 – $\square$ = 4.991 ...

Using Number Properties

Don't use calculators for the fraction problems.

Examples: Work out: 88 001 – $\square$ = 84 067 1 000 000 – $\square$ = 890 023

0.003456 – $\square$ = 0.0000312 $10\frac{3}{7} - \square = 1\frac{1}{7}$ $\frac{67}{77} - \square = \frac{3}{77}$ $3\frac{1}{2} - \square = 1\frac{1}{2}$

$23\frac{1}{11} - \square = 13$ ...

Understanding Number Properties: $a - \square = b$. What number goes in the box? ***(Answer: a – b is the number that goes in the box.)***

When Subtraction Becomes Addition

I am learning to solve problems like $\square$ – 7 566 = 13 987.

Equipment: Strips of paper. Calculators.

EA
AA
AM
AP

Using Materials

Problem: "Ana goes Christmas shopping. She spends $345.60 on toys for her children and comes home with $208.95. How much money did she have to start with?"

Write $\square - 345.60 = 208.95$ on the board. Have the students cut a strip of paper, representing Ana's unknown starting amount.

Cut off a piece and write $345.60 on it.

	$345.60

Discuss what goes on the other piece. *(Answer: $208.95.)*

Discuss what the starting amount must have been. *(Answer: $345.60 + $208.95 = $554.55.)*

Examples: Word stories and recording for: $\square - 345 = 789$ $\square - 456 = 102$

$\square - 67.9 = 43.19$ $\square - 67 = 90$ $\square - 1\,000 = 3\,000$ $\square - 200 = 560$ ...

Using Imaging

Examples: Imagine the strips to solve these problems: Word stories and recording for:

$\square - 200 = 600$ $\square - 220 = 230$ $\square$ – \$7.90 = \$3.10 $\square - 111 = 238$ $\square - 10\,000 = 4\,500$ ...

Using Number Properties

Examples: Don't use calculators for the fraction problems. Word stories and recording for:

$\square - 212.98 = 600.0034$ $\square - 200.08 = 45.89$ $\square - 7.9909 = 3.1091$ $\square - 10 = \frac{3}{4}$

$\square - 2\frac{1}{3} = \frac{1}{3}$ $\square - 0.04 = 0.0009$ ...

Understanding Number Properties: *a* and *b* stand for any numbers: $\square - a = b$. How would you work out the number to put in the box? ***(Answer: I would work out the answer to a + b.)***

Checking Addition and Subtraction by Estimation

Good number sense skills are required to "sense" when answers are wrong. This is often achieved by doing estimates.

I am improving my number sense by learning to make estimations *after* addition and subtraction calculations.

Equipment: Worksheet (Material Master 8–1).

Using Number Properties

Problem: "Julian adds 34 567 and 478 on his calculator and gets 34 089."

Write 34 567 + 478 = 34 089 on the board. Discuss why Julian must be wrong.

(Answer: The answer had to be bigger than 34 567 because he is adding.)

Discuss what Julian did wrong.

(Answer: Probably he pressed the subtraction button rather than the addition button.)

Problem: "Julian adds 34 567 and 478 on his calculator and this time gets 35 045. Check this answer by estimation."

(Possible answer: 567 + 478 is a bit over 1 000 so the answer must be a bit over 35 000. So Julian's answer looks reasonable. So Julian accepts the calculator answer.)

Examples: Worksheet (Material Master 8–1).

Understanding Number Properties: Make up a five-digit plus five-digit subtraction of your own and explain how you would estimate the answer.

EA
AA
AM
AP

Learning Experiences for Advanced Multiplicative–Early Proportional Part-Whole

Activities from Other Books

Cut and Paste
Page 49, Book 6.
I am learning to solve multiplication problems using doubling and halving, and thirding and trebling.

Multiplication Smorgasbord
Page 52, Book 6.
I am learning to solve multiplication problems using a variety of mental strategies.

Proportional Packets
Page 54, Book 6.
I am learning to solve division problems using how many times one number will go into another.

The Royal Cooking Lessons
Page 57, Book 6.
I am learning to solve division problems by changing them to simpler problems that have the same answer.

Remainders
Page 60, Book 6.
I am learning to solve division problems that have remainders.

Paper Power
Page 63, Book 6.
I am learning to work out multiplication and division problems using written working forms.

Cross Products
Page 67, Book 6.
I am learning to multiply multi-digit whole numbers.

Pipe Music with Decimals
Page 22, Book 7. (2007 edition)
I am learning to add and subtract decimals.

Deci-mats
Page 25, Book 7.
I am learning to identify equivalent fractions and to name fractions as decimals.

Candy Bars
Page 27, Book 7.
I am learning how the numbers on the left and right of the decimal point contribute to the size of the number and how decimals can be added and subtracted.

New Activities

Reversals with Multiplication and Division

I am learning how to do reversals with multiplication and division to solve problems like $263 \times \square = 456$.

Equipment: Calculators. Worksheet (Material Master 8–2).

Using Number Properties

Problem: “Snapper costs $13.89 per kilogram. Maru bought some snapper, and it cost him $56.80. When he got home, he wondered what weight of snapper he had bought.”

Discuss the problem and write it on the board as $13.89 \times \square = 56.80$.

Discuss how to solve such problems and how to round the calculator answer sensibly.

(Answer: $13.89 \times \square = 56.80$ is the same as $\square = 56.80 \div 13.89 = 4.089272858$. As this is kilograms, it makes no sense to have an answer more accurate than the nearest gram, that is to say, the nearest 0.001 kilogram. So rounding to 3 decimal places gives 4.089 kilograms.)

Problem: “Filbert had 7.475 kilograms of olives. He put a set weight of olives in a set of jars and found he had enough for 13 jars. Write this down as a division problem.” *(Answer: $7.475 \div \square = 13$.)* “Later Filbert wanted to add labels showing the weight of olives in each jar, but he had forgotten what this weight was. How would Filbert solve $7.475 \div \square = 13$?”

(Answer: $7.475 \div \square = 13$ is the same as $\square = 7.475 \div 13 = 0.575$ grams.)

Examples: Worksheet (Material Master 8–2).

Understanding Number Properties: Choose one answer:

If $a \times \square = b$	$\square$ equals	$a \div b$	$b \div a$	$a \times b$	(Answer: $b \div a$)
If $a \div \square = b$	$\square$ equals	$a \div b$	$b \div a$	$a \times b$	(Answer: $a \div b$)
If $\square \div a = b$	$\square$ equals	$a \div b$	$b \div a$	$a \times b$	(Answer: $a \times b$)
If $\square \times a = b$	$\square$ equals	$a \div b$	$b \div a$	$a \times b$	(Answer: $b \div a$)

Checking Multiplication by Estimation

I am improving my number sense by learning to check the answers for whole number multiplication problems by estimation.

Equipment: Worksheet (Material Master 8–3).

Using Number Properties

Problem: "Marcia and Linda swap books to mark each other's work. Marcia has written $32 \times 78 = 2\,574$. Linda looks at the 4 in the answer and immediately says the answer is wrong. How does Linda know?"

(Answer: $2 \times 8 = 16$ so the answer must have a 6 in the ones column.)

"Marcia sees Linda has written $32 \times 78 = 2\,856$. While the ones digit is correct, Marcia says Linda must be wrong. How does Marcia know?"

(Possible answer: $32 \times 78 \approx 30 \times 80 = 2\,400$. So 2 856 is well away from the correct answer.)

Problem: Discuss rounding 25.45×34.01.

(Possible answer: Here the problem is that rounding both numbers up to 30×40 respectively gives the answer 1 200, which is too large, and rounding down to $20 \times 30 = 600$ is too low. Somewhere between 600 and 1 200, say 900, is sensible.)

Problem: Discuss why 29.023×88.912 is a little less than 2 700.

(Possible answer: 29.023 is a little less than 30 and 88.912 is a little less than 90.)
So $29.023 \times 88.912 < 30 \times 90 = 2\,700$.

Examples: Worksheet (Material Master 8–3). Do *not* use calculators.

Understanding Number Properties: Make up a four-digit decimal number times a three-digit decimal number problem of your own and explain how you would estimate the answer.

Checking Division by Estimation

I am improving my number sense by learning to check the answers for whole number division problems by making estimations.

Equipment: Worksheet (Material Master 8–4).

Using Number Properties

Problem: "The manager at the bookstore buys 69 books for $1,327. Estimate the price of one book."

Discuss why 69 is rounded to 70. *(Answer: 70 has only 1 significant figure so tables will be able to be used in the next step.)*

Write $1\,327 \div 70$ on the board and discuss what 1 327 could be rounded to. *(Possible answer: Rounding to say 1 300 or even 1 000 produces difficult divisions to do mentally. But rounding 1 327 to 1 400 produces the division $1\,400 \div 70 = 20$.)*

Examples: Worksheet (Material Master 8–4).

Understanding Number Properties: Make up a four-digit decimal number divided by a two-digit decimal number problem of your own and explain how you would estimate the answer.

To Turn or Not to Turn

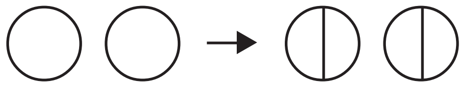
Many students try turning division problems around when this is inappropriate.
For example, $4 \div 20$ is seen as impossible, so students turn it around to $20 \div 4$ and get 5.

I am learning that changing the order in division matters.

Equipment: Fraction pieces.

Using Materials

Problem: "Make up word problems for each of these divisions: $4 \div 2$ and $2 \div 4$, and solve them. Use fraction pieces to help you."



(Answer: Jane has 4 chocolates to share between 2 people. So $4 \div 2 = 2$. And Kiri has 2 chocolates to share among 4 people. So $2 \div 4 = \frac{1}{2}$.)

Examples: Repeat for: $6 \div 2, 2 \div 6$ $6 \div 3, 3 \div 6$ $12 \div 2, 2 \div 12$ ...

Using Number Properties

Examples: Examine the pattern in the pairs of answers above then work out these:
$15 \div 3, 3 \div 15$ $100 \div 10, 10 \div 100$ $60 \div 3, 3 \div 60$

Understanding Number Properties: If the answer to $a \div b$ is c, what is the answer to $b \div a$? ***(Answer:** $\frac{1}{c}$.)*

The Equals Sign Again

In *Book 5: Teaching Addition, Subtraction, and Place Value*, page 40, the following comment about the equals sign appears:

> "Many students regard the '=' sign as meaning only 'get the answer'. That is what 'equals' means on a calculator. But the notion of equals is more general than this. In the problem below, the equals sign means that the total on the left of the sign is the same as the total on the right."

This more general meaning of equals is extended into multiplication and division.

I am learning that the answer on the left of the equals sign is the same as the answer on the right of the equals sign.

Equipment: None.

Using Number Properties

Problem: "Jocelyn knows 12×12 is 144, and she knows 2 is a factor of 144 because 144 is even. Complete this statement: $12 \times 12 = 2 \times \square$."

Discuss the balance idea of equals; the left-hand side is 144, so the right-hand side is 144.
And $2 \times 72 = 144$, so $\square$ is 72.

Examples: Complete these equations: $6 \times 10 = 3 \times \square$ $\square \times 20 = 8 \times 10$
$60 \div \square = 20 \div 4$ $\square \div 4 = 30 \div 2$ $6 \times \square = 3 \times 40$ $6 \times 10 = 3 \times \square$ $2 \times 4 + 3 = \square + 5$
$2 \times 4 + \square = 11$ $2 \times \square = 11 - 3$ $\square - 3 = 11 + 1$

Understanding Number Properties: Find two numbers that go in the boxes to make this equation correct: $35 \div 5 = \square \div \square$.

Using 0

Intuitively many students (and adults) think division by 0 produces an answer of 0. This is understandable because division by 0 occurs so seldom in real world situations.

I am learning that 0 divided by any number is 0, and that I can't divide by 0.

Equipment: None.

Using Number Properties

Problem: "Make up a word problem for $0 \div 34$ and solve it."

(Possible answer: Marie thinks about dividing nothing into 34 pieces. Answer: 0.)

Examples: Work out: 0×456 $\quad 0 \div 89$ $\quad 89 \times 0$ $\quad 0 \div 789\ 093$ $\quad 345 \times 567 \times 0$

$0 \div 834 \div 2\ 345$

Understanding Number Properties: If letters stand for any numbers, find:
$a \times b \times 0$ $\quad 0 \div a$ $\quad a \times 0 \div b$ $\quad a \times b \times c \div d \times 0$

Hard Problem: "Find $34 \div 0$ on a calculator. What does the "E" in the display indicate? Why is division by 0 impossible?"

(Brief answer: Consider this problem: There is a queue of people. Jill has 34 apples and she gives out 0 apples to each person in the queue. So in a sense, she can give out apples to an infinite number of people. So $34 \div 0 = \infty$. Because mathematicians have a strong aversion to saying $34 \div 0 = \infty$, they prefer to say $34 \div 0$ is undefined.

Order of Operations

Scientific calculators and spreadsheets produce correct answers to multiple-step calculations as they are programmed to follow the correct conventions. Some care is required with these conventions. For example, there is a "left to right" convention to work out $12 – 6 – 4$ giving 2 whereas $12 – (6 – 4)$ overrides this convention to produce 10. This needs careful teaching.

I am learning to calculate using the conventional order of operations.

Equipment: Simple four-function calculators. Scientific calculators or access to spreadsheets on computers. Worksheet (Material Master 8–5).

Using Number Properties

Problem: "Julia's simple calculator works out $8 + 3 \times 6$ and gets 66. Maree works out $8 + 3 \times 6$ on a scientific calculator or a spreadsheet and gets 26. Discuss why the answers are different."

(Answer: Simple calculators work from left to right through a multi-stepped calculation. So $8 + 3$ produces 11 on screen, then $\times 6$ gives the answer to 11×6 which is 66. But scientific calculators and spreadsheets are programmed to do the multiplication first. So for $8 + 3 \times 6$, they work out 3×6 first to get 18. Then they work out $8 + 18$ to get 26.)

Problem: "Julia buys 3.456 kg of potatoes on Monday and 2.087 kg of potatoes on Tuesday. She writes the cost down as $3.456 + 2.087 \times 0.91$. Why will this not be correct on a scientific calculator?"

Potato Special $0.91 per kg

(Answer: The addition needs to be done first in this problem.)

Discuss the effect of brackets. Write $3.456 + 2.087 \times 0.91$ on the board and add brackets. *(Answer: $(3.456 + 2.087) \times 0.91$.)*

Get the students to enter the numbers and brackets on a scientific calculator or spreadsheet and get them to answer to the nearest cent. Discuss whether the answer is correct. *(Answer: It is correct. Brackets override the "times before add" rule.)*

Problem: "Work out $(2 + 3 \times 4) \times 2$ without a calculator."

Discuss the convention that brackets are calculated first, and *within* brackets, the times before add convention is obeyed. *(Answer: $(2 + 3 \times 4) \times 2 = (2 + 12) \times 2 = 14 \times 2 = 28$.)*

Problem: "Work out $(2 + 6 \div 3) \times 2 – 14 \div 7$ without a calculator."

Introduce the convention that, while brackets are still calculated first, within brackets, multiplication/division precede addition/subtraction. *(Answer: 6.)*

Problem: "Work out (44 – 8 ÷ 2) ÷ (14 + 3 – 15) + (4 ÷ 2 × 4) without a calculator."

Introduce the convention that, within brackets, if the operations are only + and –, work through the bracket from left to right because + and – are equal in importance. Similarly if the bracket contains only × and ÷, work left to right through the bracket as × and ÷ are equal in importance.

Examples: Worksheet (Material Master 8–5).

$$(44 - 8 \div 2) \div (14 + 3 - 15) + (4 \div 2 \times 4)$$
$$(44 - 4) \div 2 + 8$$
$$40 \div 2 + 8$$
$$20 + 8$$
$$28$$

Doubling and Halving

This revises and extends the activity Cut and Paste, page 25, *Book 6: Teaching Multiplication and Division*.

I am extending applications of halving and doubling, thirding and trebling for multiplication and division.

Equipment: Worksheet (Material Master 8–6).

Using Number Properties

Problem: Write 24 on a card and stick it on the board. Write ÷ 2 beside it and ask the answer. Write × 2 beside it and ask the answer. So 24 ÷ 2 × 2 = 24 is on the board.

Examples: Repeat for 10 × 2 ÷ 2 100 × 2 ÷ 2 6 ÷ 3 × 3 ...

Understanding Number Properties: Make up an example that shows the effect of multiplying and dividing by the same number. ***(Answer: The original number stays the same.)***

Problem: Write 5 × 468 024 806 642 = 2 340 124 033 210 on the board. Ask the students how this works.

(Answer: Multiplying 5 by 2 gives 10 and dividing 468 024 806 642 by 2 gives 234 012 403 321. And 10 × 234 012 403 321 = 2 340 124 033 210.)

Problem: "Find $3\frac{1}{3} \times 3\,636$."

Write $3 \times 3\frac{1}{3}$ on the board and discuss the answer. *(Answer: 10.)*

Discuss why $3\frac{1}{3} \times 3\,636 = 10 \times 1\,212 = 12\,120$.

Problem: "Which is larger 30 ÷ 5 or 60 ÷ 10 or are they both the same?" Discuss why they are the same.

Problem: "Does 340 ÷ 0.5 = 680 ÷ 1? How does this help to find 340 ÷ 0.5?"

Examples: Worksheet (Material Master 8–6).

Understanding Number Properties: Does 25 × 36 = (25 × 89) × (36 ÷ 89)? Explain. ***(Answer: Yes. Multiplying and dividing by the same number balances out and leaves the original problem the same.)***

Multiplying by 25

I am learning a quick way to multiply by 25.

Equipment: Place-value material, for example play money.

Using Materials

Problem: "Tana collects $25 from each person in a group of 9 people to buy concert tickets. How much money does he have?"

Get the students to lay out 9 piles of $25 in play money. The students look at the first 4 piles. How much money is in the piles? *(Answer: $100.)*

"Could you make another pile of $100?"

(Answer: Yes, because there are 5 piles left, and we need only 4 to make $100.)

Discuss why the total is $225.

Examples: Word stories and recording for: 7×25 5×25 12×25 10×25 14×25 ...

Using Imaging

Problem: "Tana collects $25 from each person in a group of 11 people to buy concert tickets. How much money does he have?"

Shield 11 piles of $25 from the students. Ask them to imagine how the bundles will be combined to form $100 piles. If necessary, revert to materials by showing the piles.

Examples: Word stories and recording for: 21×25 24×25 16×25 15×25 13×25 ...

Using Number Properties

Problem: "Work out 25×35."

Discuss reversing the problem to 35×25. Discuss how many groups of 4 are in 35. *(Answer: 8.)*

"What is the remainder?" *(Answer: 3.)*

Discuss how this leads to the answer being 875.

Examples: Work out: 40×25 25×25 25×41 80×25 25×45

Understanding Number Properties: Make up your own problem of 25 times a two-digit number and explain how to solve it.

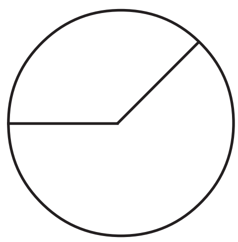
Estimating with Fractions

An important part of number sense is to have a concept of the size of fractions independent of doing calculations. For example, a student with a good sense of number will know immediately that $\frac{11}{21}$ is more than $\frac{1}{2}$ because 11 is over half the available 21 pieces.

I am using my number sense to estimate the answers to addition and subtraction of fractions without calculating the exact answer.

Equipment: Two different-coloured circle pieces that are rotatable to show angles (Material Master 8–7).

Using Materials



Problem: Show $\frac{4}{9}$ using the two different-coloured circle pieces and discuss whether it is more or less than $\frac{1}{2}$.

Each student rotates the circles until one colour circle shows as near to $\frac{4}{9}$ as they can. Discuss why it is $\frac{4}{9}$.

Look at the other colour and discuss what fraction it is.

(Answer: $\frac{5}{9}$ because there are $\frac{9}{9}$ in 1 whole.)

Examples: Show these fractions on the circles and work out the other fraction that adds up to 1 whole: $\frac{3}{7}$ $\frac{4}{5}$ $\frac{5}{9}$ $\frac{5}{11}$ $\frac{11}{21}$ ...

Using Imaging

Problem: "I have made $\frac{15}{31}$, but you cannot see it. Describe in words what this fraction looks like." *(Possible answer: the fraction is just a little bit less than $\frac{1}{2}$.)*

"What is the other fraction?" *(Answer: $\frac{16}{31}$.)*

Examples: Describe these fractions without using the circles and work out the other fraction that adds up to 1 whole: $\frac{5}{11}$ $\frac{2}{5}$ $\frac{4}{7}$ $\frac{8}{17}$ $\frac{6}{13}$

Using Number Properties

Problem: "Maurice eats $\frac{2}{5}$ of a cake, and Norris eats $\frac{3}{7}$ of a same sized cake. In total, do they eat more or less than 1 cake?"

Discuss the answer. *(Answer: Both are a little less than $\frac{1}{2}$ so the total is less than 1 whole.)*

Examples: Without calculating, determine whether these are more or less than 1:
$\frac{2}{5} + \frac{12}{23}$ $\frac{17}{33} + \frac{6}{11}$ $\frac{51}{100} + \frac{6}{12}$ $\frac{24}{48} + \frac{29}{60}$ $\frac{24}{47} + \frac{61}{120}$ ...

Examples: Is $1 - \frac{21}{43}$ more or less than $\frac{1}{2}$? Is $\frac{11}{45}$ more or less than $\frac{1}{4}$? Is $\frac{3}{4} - \frac{11}{40}$ more or less than $\frac{1}{2}$? Is $\frac{3}{4} + \frac{15}{62}$ more or less than 1?

Hard example: Is $\frac{3}{7} + \frac{14}{27}$ more or less than 1? *(Answer: Here the gap between $\frac{3}{7}$ and $\frac{1}{2}$ is more than the gap between $\frac{14}{27}$ and $\frac{1}{2}$. So $\frac{3}{7} + \frac{14}{27} < 1$.)*

Understanding Number Properties: Make up two fractions that both have denominators greater than 30 and that add up to just a little bit less than 1.

Fractions

I am using my number sense to order fractions.

Equipment: None.

Using Number Properties

Problem: "List all the fractions less than 1 whole that can be made using only 3, 7, and 11." *(Answer: $\frac{3}{7}$, $\frac{3}{11}$, $\frac{7}{11}$.)*

Discuss which of those fractions is the smallest ($\frac{3}{11}$) and the largest ($\frac{7}{11}$).

Examples: Make the smallest fraction possible from two of these numbers:
3, 5, 17 10, 50, 61 30, 7, 2 ...

Examples: Find the closest fraction to possible from two of these numbers:
3, 5, 17 10, 50, 61 30, 7, 2 ...

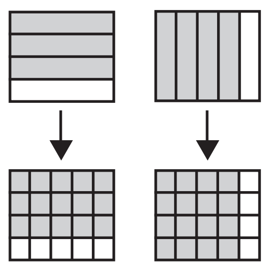
Understanding Number Properties: Make up a fraction that has a denominator greater than 100 and that is just a little bit more than $\frac{1}{2}$.

Equivalent Fractions

After understanding what a fraction is, equivalent fractions is the next most important concept in fractions. It is needed in all the operations on fractions. In this activity, the fraction shapes are rectangles rather than circles as rectangles are much easier to draw.

I am learning how to convert a fraction to any equivalent fraction.

Equipment: Sheets with a 2-centimetre grid (Material Master 8–8). Worksheet (Material Master 8–9).



Using Materials

Teaching the comparison of fractions requires considerable care.

Problem: "Identical twins Ronald and Donald get identical cakes for their birthdays. Ronald cuts his cake into 4 equal pieces and eats $\frac{3}{4}$ of it. Donald cuts his cake into 5 equal pieces and eats $\frac{4}{5}$ of it. Who has eaten more cake?"

Get the students to draw the cakes as two 8 centimetre by 10 centimetre rectangles using Material Master 8–8. Add horizontal lines to Ronald's cake and vertical lines to Donald's cake. Discuss why it is easier to add horizontal lines to Ronald's cake and vertical lines to Donald's cake.

(Answer: Horizontal lines follow the grid lines for quarters, but they don't for fifths. Fifths follow the vertical grid lines.)

EA
AA
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AP

Discuss why adding vertical and horizontal lines respectively enables you to compare the shaded areas. *(Answer: Each cake now has 20 parts.)*

Discuss why for Ronald $\frac{3}{4} = \frac{15}{20}$ and why for Donald $\frac{4}{5} = \frac{16}{20}$.

Discuss how this shows $\frac{3}{4} < \frac{4}{5}$. *(Answer: $\frac{15}{20} < \frac{16}{20}$ from the diagrams.)*

Problem: "Draw a pair of 8 centimetre by 6 centimetre rectangles on the supplied grid. (See www.nzmaths.co.nz)
Divide each rectangle into 12 squares. Show $\frac{3}{4}$ on one grid and $\frac{2}{3}$ on the other.
Decide whether $\frac{3}{4} > \frac{2}{3}$ or $\frac{2}{3} > \frac{3}{4}$."

(Answer: $\frac{9}{12} > \frac{8}{12}$ so, $\frac{3}{4} > \frac{2}{3}$.)

Examples: Decide which fraction is the larger by drawing rectangles. Carefully plan the size of the rectangles before starting drawing: $\frac{4}{5}$ or $\frac{5}{6}$ $\quad$ $\frac{1}{3}$ or $\frac{3}{7}$ $\quad$ $\frac{5}{6}$ or $\frac{6}{7}$ ...

Using Imaging

Problem: "From the examples and problems above, Donna summarises the data and looks for patterns. What patterns can you see?"

$\frac{4}{5} = \frac{16}{20}$	$\frac{3}{4} = \frac{15}{20}$
$\frac{3}{4} = \frac{9}{12}$	$\frac{2}{3} = \frac{8}{12}$
$\frac{4}{5} = \frac{24}{30}$	$\frac{5}{6} = \frac{25}{30}$
$\frac{6}{7} = \frac{36}{42}$	$\frac{5}{6} = \frac{35}{42}$

Problem: "Donna wonders how to solve $\frac{3}{7} = \frac{\square}{28}$.

Imagine what a rectangle with horizontal lines showing $\frac{3}{7}$ would look like. What vertical lines would be needed to cut the whole into 28 equal pieces? Answer the problem $\frac{3}{7} = \frac{\square}{28}$." Fold back to drawing a picture if needed.

Examples: Imagine pictures and complete these: $\frac{3}{4} = \frac{\square}{16}$ $\quad$ $\frac{2}{3} = \frac{\square}{15}$ $\quad$ $\frac{5}{6} = \frac{\square}{18}$ $\quad$ $\frac{3}{10} = \frac{\square}{20}$ $\quad$ $\frac{7}{9} = \frac{\square}{27}$.

Using Number Properties

Examples: Worksheet (Material Master 8–9).

Understanding Number Properties: Make up a fraction that has a denominator greater than 99 and that equals $\frac{4}{5}$.

Fractions Greater than 1

Students often believe fractions must be less than 1 because they are thinking about a fraction of 1 whole. A more general view of fractions is needed. This activity revisits this important idea.

I am learning to convert mixed fractions into common fractions and vice versa.

Equipment: Fraction pieces.

Using Materials

Problem: "Kate's Kitchen sells pizza pieces. Each pizza is cut into 4 equal pieces. At the end of the day, the computer shows Kate has sold 13 pieces of pizza. How much pizza has she sold?"

(Answer: Modelling pizza with 13 quarters, and forming them into 3 wholes and 1 quarter shows $\frac{13}{4} = 3\frac{1}{4}$.)

Examples: Using word stories convert these fractions to mixed fractions: $\frac{15}{4}$ $\quad$ $\frac{11}{3}$ $\quad$ $\frac{12}{4}$ $\quad$ $\frac{17}{6}$ $\quad$ $\frac{19}{3}$ $\quad$ $\frac{24}{4}$ ...

Using Imaging

Problem: "Imagine 19 pieces of pizza where each piece is a fifth of a pizza. How many pizzas is this?"

Write $\frac{19}{5}$ = on the board. Have the students discuss how this is $3\frac{4}{5}$.

Examples: Using word stories, convert these fractions to mixed fractions. Imagine pieces of pizza: $\frac{15}{7}$ $\quad$ $\frac{11}{6}$ $\quad$ $\frac{30}{6}$ $\quad$ $\frac{27}{10}$ $\quad$ $\frac{19}{4}$ $\quad$ $\frac{24}{7}$ ...

EA
AA
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AP

Using Number Properties

Problem: "At a party, Jules gives each of his guests 1 tenth of a cake. He has 112 guests. How many cakes are eaten?" *(Answer: $11\frac{2}{10}$.)*

Examples: Using word stories convert these fractions to mixed fractions: $\frac{101}{10}$ $\frac{31}{3}$ $\frac{99}{11}$ $\frac{52}{5}$ $\frac{41}{8}$ $\frac{103}{20}$ …

Problem: "Moana has $10\frac{1}{6}$ pizzas in her fridge for a party. She plans to give $\frac{1}{6}$ of a pizza to each guest. How many guests will get a piece of pizza?"

Discuss why the answer is $10 \times 6 + 1 = 61$, and write $10\frac{1}{6} = \frac{61}{6}$ on the board.

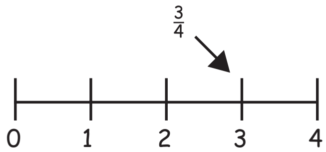
Examples: Convert these fractions to common fractions: $3\frac{1}{10}$ $7\frac{2}{3}$ $1\frac{23}{24}$ $12\frac{2}{5}$ $6\frac{7}{8}$ $3\frac{5}{12}$ …

Understanding Number Properties: Describe how to convert $120\frac{4}{5}$ to a common fraction.
(Answer: Multiply the whole number (120) by the denominator (5) and add to the numerator (4), the answer 604 is the numerator, 5 is the denominator. So $120\frac{4}{5} = \frac{604}{5}$.)

Understanding Number Properties: Describe how to convert $\frac{209}{5}$ into a mixed fraction.
(Answer: Divide the numerator (209) by the denominator (5), this gives the whole number part of the mixed fraction, 41. Put the remainder as the numerator over the denominator, $\frac{4}{5}$. So $\frac{209}{5} = 41\frac{4}{5}$.)

Fraction Number Lines

Students who do not understand fractional numbers well have trouble drawing number lines showing fractions. For example, this is a common wrong answer to a request to place $\frac{3}{4}$ on a number line:



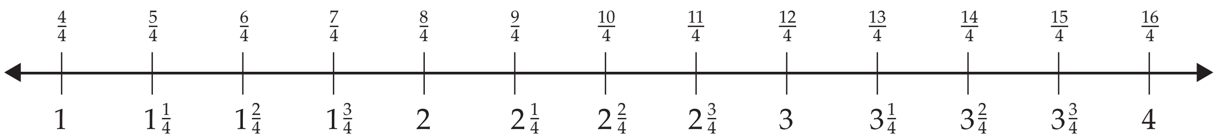
I am learning to show mixed fractions and common fractions on double number lines.

Equipment: A set of blank double number lines (Material Master 8–10).

Using Materials

Problem: "$\frac{\square}{4}$ is a number between 2 and 4. What numbers could go in the box?"

Get the students to draw and mark a double number line in both mixed fractions and common fractions.



Then answer the question.

(Answer: $\frac{9}{4}$, $\frac{10}{4}$, $\frac{11}{4}$, $\frac{12}{4}$, $\frac{13}{4}$, $\frac{14}{4}$, $\frac{15}{4}$ are all between 2 and 4.)

Example: "$\frac{\square}{3}$ is a number between 3 and 5. What numbers could go in the box?"

Using Imaging

Problem: "$\frac{\square}{5}$ is a number between $3\frac{2}{5}$ and $3\frac{4}{5}$. Imagine which number could go in the box." Fold back to drawing the number line if necessary.

(Answer: Only $3\frac{3}{5}$ fits between $3\frac{2}{5}$ and $3\frac{4}{5}$. And $3\frac{3}{5} = \frac{18}{5}$. So 18 goes in the box.)

Examples: Find the number that can go in the box:

$\frac{\square}{5}$ is between $2\frac{1}{5}$ and $2\frac{3}{5}$. $\frac{\square}{7}$ is between $1\frac{4}{7}$ and $1\frac{6}{7}$.

Using Number Properties

Examples: Find the number that goes in the box:

$\frac{\square}{5}$ is between $3\frac{3}{5}$ and 4. $\frac{\square}{3}$ is between 8 and $8\frac{2}{3}$.

$3\frac{\square}{6}$ is between $\frac{20}{6}$ and $\frac{22}{6}$. $3\frac{\square}{6}$ is between $\frac{21}{6}$ and $\frac{23}{6}$.

Scales on Number Lines

When drawing a number line to show, say, 3.45 with 3.4 and 3.5 as the end points, students easily lose track of the position of 0 on the line. Being able to solve such a problem is a powerful indicator that the students' knowledge of the decimal place value system is good. This activity is not easy and needs careful teaching.

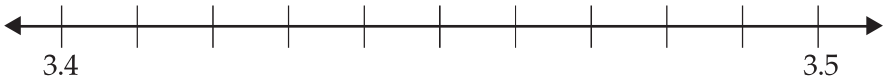
I am learning how different scales are needed to show decimal numbers on number lines.

Equipment: Metre rules.

Using Materials

Problem: "If 3.4 and 3.5 are marked 10 divisions apart on a number line, where is the number 0?"

Get the students in groups to mark 3.4 and 3.5 on a number line with 10 1-centimetre gaps between the 2 numbers.



Discuss why the distance from 0 to 3.4 is 34 × 10 centimetres. Get the students to use a metre rule to locate 0.

Problem: "Outside place 2 pegs, representing 4.5 and 4.6 on a number line, 1 pace apart. Your task is to locate where 0 would be on the number line."

Put the students into groups of 3 or 4 and discuss what they are going to do outside on the field. Show them the task on the board. They will put a marker on the ground to represent 4.5. One student in each group takes a pace to represent 4.6, and this point is marked with 4.6. The students' task now is to mark where 0 is. (The group moves 45 paces back from 4.5 using the pace of the selected student.)

Examples: The space between the given numbers represents one pace. Find 0 in each case:
2.2 to 2.3, locate 0. *(Answer: 0.1 is 1 pace, so they need 22 paces back from 2.2.)*

0.45 to 0.46, locate 0. *(Answer: 0.01 is 1 pace, so they need 45 paces back from 0.45.)*

0.023 to 0.024, locate 0. *(Answer: 0.001 is 1 pace, so they need 23 paces.)*

150 to 160, locate 0. *(Answer: 10 is 1 pace, so 150 is 15 paces.)*

3 300 to 3 400, locate 0. *(Answer: 100 is 1 pace, so 3 300 is 33 paces.)*

Using Number Properties

Examples: 2.02 to 2.03 are 10 centimetres apart on a number line. How far is it to 0?

*(Answer: 0.01 unit = 10 centimetres so 1 unit is 100 × 10 centimetres = 10 metres.
So 2.02 is 2.02 × 10 = 20.2 metres back to 0.)*

67.5 to 67.6 are 1 metre apart on a number line. How far is it to 0?
907.05 to 907.06 are 10 centimetres apart on a number line. How far is it to 0?

Whole Number Rounding

Whole number rounding is a gentle yet important introduction to using number lines to round decimal numbers sensibly.

I am learning to round numbers to the nearest 1, 10, 100, and 1 000.

Equipment: Marked/unmarked number lines (Material Master 8–11).

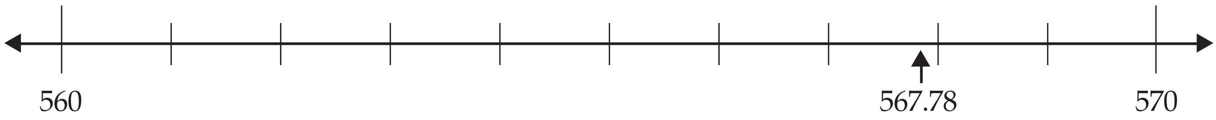
Using Materials

Problem: "Betty's Appliance Store prices TVs to the nearest $10. If the exact price is $567.78, what does Betty charge?"

EA
AA
AM
AP

Discuss why the choices are 560 and 570 and enter these at the end of the number line 10 units apart.

Discuss why 567 is 7 divisions along from 560. Discuss why 567.78 is nearer 568 than 567 and locate 567.78 with an arrow on the number line.



Discuss why 567.78 ≈ 570 to the nearest 10.

Examples: Stories for rounding on number lines:

$456.89 to the nearest $10 $2,345.09 to the nearest $10
$7,456 to the nearest $100 $24,509 to the nearest $1,000

Using Imaging

Problem: "A builder works out the price of building a house as $123,566.89. He quotes a price to the nearest $1,000. What is his price?"

Discuss the two alternatives ($123,000 and $124,000). Ask the students to draw an *empty* number line (Material Master 8–11) and put 123 000 and 124 000 at the ends. Ask where 123 500 would be on the number line. *(Answer: Half way.)*

So is 123 566.89 closer to 123 000 or 124 000? *(Answer: 124 000. So the builder charges $124,000.)*

Examples: Stories for rounding on empty number lines:

$345,902.89 to the nearest $1,000 $266,345.09 to the nearest $100
$794,856.23 to the nearest $10,000 $5,624,509 to the nearest $100,000

Using Number Properties

Examples: Stories for rounding *without* number lines:

9 099.99 to the nearest 10 795.078 to the nearest 100
94 756.2309 to the nearest 10 000 999.999 to the nearest 1
11.89 to the nearest 1 10 890.8 to the nearest 100
956.2709 to the nearest 10 999 999.9 to the nearest 1

Understanding Number Properties: A number rounds to 345 800 to the nearest hundred. What kind of number could it be?

Confusing Fractions and Decimals

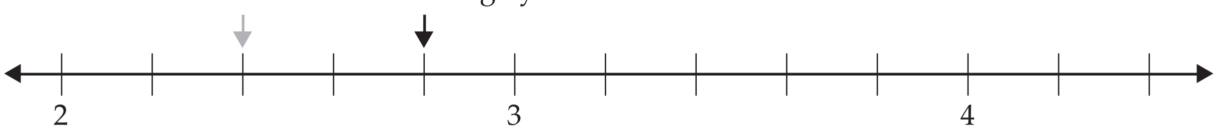
It is easy for students with poor understanding of decimals to mistakenly think locating say 2.4 on a number line is simply a matter of counting along 4 divisions from 2 towards 3.

> I am demonstrating my number sense by showing the difference between decimal and fraction number lines.

Equipment: Worksheet (Material Master 8–12).

Using Materials

Problem: "Jolene draws this number line on the board and adds an arrow where she reckons 2.4 is located (the black arrow). Explain Jolene's reasoning. Why is Jolene wrong?" Draw Jolene's number line on the board without the grey arrow for the moment.



Discuss why Jolene is wrong. *(Answer: Counting along 4 divisions only works if there are 10 divisions not 5 as in this case.)*

Discuss where 2.4 actually is. *(Answer: Imagine 10 divisions and add the grey arrow.)*

Examples: Worksheet (Material Master 8–12).

Rounding Decimals

The common rounding rule that "if the digit is over 5, go up" is in fact not always correct. Successive rounding of each decimal place in turn can lead to errors. For example, to round 4.48 to the nearest whole number, a common incorrect application of the rule is this: Round 4.48 to 4.5, round 4.5 to 5. Yet 4.48 is 4 to the nearest whole number. A reliable method for rounding is this: select the possible rounded number below and the possible rounded number above. See which of these the number to be rounded is closer to.

I am learning a sense of rounding decimal numbers.

Equipment: Worksheet (Material Master 8–14).

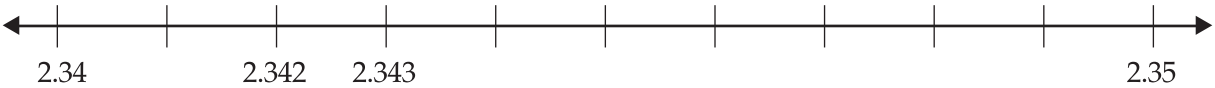
Using Materials

Problem: "Place 2.3429678335 on a number line between 2.34 and 2.35 as accurately as possible."

Write 2.3429678335 on the board and draw a number line with 10 divisions on the board as wide as the board. (The divisions need only be roughly equal, as measuring them will take too long. (All the better if the board is the width of the room!)

Discuss where 2.342 is. Draw 10 divisions between 2.342 and 2.343. Locate 2.3429.

Continue locating 2.3429678335 decimal place by decimal place until the interval is too small to continue.



Examples: Worksheet (Material Master 8–14).

Using Number Properties

Examples: Round the following to a number of decimal places:

3.567 to 2 dps	12.5697 to 1 dp	109.780678 to 4 dps
0.002376 to 5 dps	10.09 to 1 dp	12.5067 to 3 dps
56.597 to 1 dp	0.0000371 to 5 dps	1.59709272 to 6 dps

Dividing Fractions

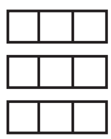
Students normally regard division as sharing. The other meaning of division is measurement, namely finding the number of equal sets that can be made. This is the most useful view for making sense of division by a fraction.

I am learning how to divide a fraction by a fraction.

Equipment: Squared paper. Rulers.

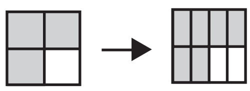
Using Materials

Problem: "Susie has 3 cakes. She intends giving her friends $\frac{1}{3}$ of a cake each. What is $3 \div \frac{1}{3}$?"



Get the students to draw rectangular cakes and divide each cake into thirds. (Circles are more complicated to work with.) Discuss why $3 \div \frac{1}{3} = 9$. So, Susie can feed 9 people.

Problem: "Malcolm has $\frac{3}{4}$ of a cake. He gives guests $\frac{1}{8}$ of a cake each. How many guests get a piece of cake?"



Write $\frac{3}{4} \div \frac{1}{8}$ on the board. Get the students to draw $\frac{3}{4}$ and add lines to create eighths. Discuss how the drawing shows $\frac{3}{4} \div \frac{1}{8} = 6$.

Examples: Draw pictures to find the answers to: $2 \div \frac{1}{4}$ $\quad \frac{2}{3} \div \frac{1}{6}$ $\quad 4 \div \frac{1}{2}$ $\quad \frac{3}{2} \div \frac{1}{8}$ ...

Using Imaging

Problem: "Describe a picture showing $\frac{3}{8}$. *Imagine* what you would do to solve $\frac{3}{8} \div \frac{1}{16}$."

Examples: Imagine pictures to find the answers to: $3 \div \frac{1}{4}$ $\frac{5}{3} \div \frac{1}{6}$ $5 \div \frac{1}{4}$ $\frac{1}{2} \div \frac{1}{6}$ ...

Using Number Properties

Problem: "Janice tries to work out $\frac{6}{8} \div \frac{1}{8}$. Reading $\frac{6}{8} \div \frac{1}{8}$ out loud as 6 eighths divided by 1 eighth, she immediately sees the answer is 6. How?"

(Answer: 6 objects divided by 1 object is always 6 regardless of what the objects are. So $\frac{6}{8} \div \frac{1}{8} = 6$.)

Examples: $\frac{9}{17} \div \frac{1}{17}$ $\frac{34}{81} \div \frac{1}{81}$ $\frac{89}{108} \div \frac{1}{108}$ $\frac{46}{3} \div \frac{1}{3}$ $\frac{67}{5} \div \frac{1}{5}$

Understanding Number Properties: What does $\frac{a}{b} \div \frac{1}{b}$ equal for any choice of a and b? ***(Answer: a, since any number divided by 1 gives that same number.)***

Problem: "Janice now tries to apply her method to harder divisions like $\frac{3}{4} \div \frac{1}{8}$. She claims $\frac{3}{4} \div \frac{1}{8} = \frac{6}{8} \div \frac{1}{8}$. And so the answer must be 6. Is she correct?" *(Answer: Yes. $\frac{3}{4} = \frac{6}{8}$.)*

Examples: Work out: $\frac{3}{4} \div \frac{1}{16}$ $\frac{5}{3} \div \frac{1}{6}$ $\frac{2}{3} \div \frac{1}{12}$ $\frac{7}{8} \div \frac{1}{4}$ $5 \div \frac{1}{4}$ ...

Understanding Number Properties: A unit fraction has 1 for the numerator. Make up a whole number divided by a unit fraction problem and solve it.

Understanding Number Properties: What does $x \div \frac{1}{z}$ equal? ***(Answer: xz.)***

Harder Division of Fractions

Dividing fractions is probably the most difficult idea students will have encountered in all their experiences with number. It requires careful teaching by the teacher and clever thinking by the students.

I am learning how to divide harder fractions.

Equipment: None.

Using Number Properties

Problem: "Juanita has $\frac{7}{8}$ of a cake. She cuts pieces of $\frac{2}{9}$ in size to put in packets for her guests. How many packets of cake will she make?" Write $\frac{7}{8} \div \frac{2}{9}$ on the board.

Discuss why it is hard to compare $\frac{7}{8}$ and $\frac{2}{9}$. *(Answer: They have different names or denominators, eighths and ninths.)*

"How could you rewrite them as equivalent fractions?" *(Answer: The lowest common multiple of 8 and 9 is 72, so you could convert eighths and ninths into "seventy-twoths". So $\frac{7}{8} = \frac{63}{72}$ and $\frac{2}{9} = \frac{16}{72}$.)*

How would you work out $\frac{63}{72} \div \frac{16}{72}$? *(Answer: $\frac{63}{72} \div \frac{16}{72} = 63 \div 16 = \frac{63}{16}$ or $3\frac{15}{16}$.)*

Examples: Find $\frac{2}{3} \div \frac{4}{7}$ $\frac{2}{7} \div \frac{11}{3}$ $\frac{10}{11} \div \frac{9}{4}$ $\frac{5}{6} \div \frac{7}{10}$

Problem: "Summarise the answers above in a table that discusses a quick way of dividing fractions."

(Answer: Flipping the second fraction and multiplying always works. So, for example $\frac{7}{8} \div \frac{2}{9} = \frac{7}{8} \times \frac{9}{2} = \frac{63}{16}$.)

$\frac{2}{3} \div \frac{4}{7} = \frac{14}{12}$	$\frac{2}{7} \div \frac{11}{3} = \frac{6}{77}$
$\frac{10}{11} \div \frac{9}{4} = \frac{40}{99}$	$\frac{5}{6} \div \frac{7}{10} = \frac{50}{42}$

Examples: Find: $\frac{4}{5} \div \frac{3}{8}$ $\frac{2}{3} \div \frac{5}{11}$ $\frac{7}{9} \div \frac{3}{5}$ $\frac{1}{4} \div \frac{5}{6}$ ...

Understanding Number Properties: What letters replace the question marks? $\frac{a}{b} \div \frac{c}{d} = \frac{? \times ?}{? \times ?}$ (Answer: $\frac{a \times d}{b \times c}$.)

Whole Numbers Times Fractions

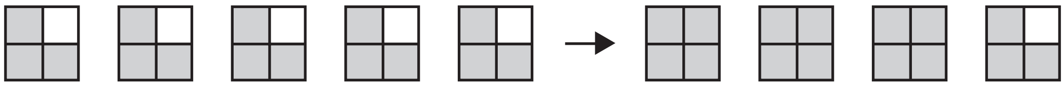
I am learning how to multiply whole numbers by fractions.

Equipment: Fraction materials.

Using Materials

Problem: "At the supermarket, 4 blocks of chocolate fit in each box. Mel notices that she has 5 boxes, each of which has 3 blocks of chocolate left in them. Mel decides to tidy up by repackaging them into whole boxes. How many boxes can Mel make up?"

Write $5 \times \frac{3}{4}$ on the board, then build 5 groups of 3 quarters with fraction material, and arrange the 15 pieces into wholes.



Discuss why $5 \times \frac{3}{4} = 3\frac{3}{4}$.

Examples: Word problems for: $3 \times \frac{3}{4}$ $\quad 3 \times \frac{2}{3}$ $\quad 4 \times \frac{2}{3}$ $\quad 5 \times \frac{4}{5}$

Using Imaging

Examples: Imagine the materials. Word problems for: $10 \times \frac{2}{3}$ $\quad 2 \times \frac{5}{7}$ $\quad 15 \times \frac{1}{5}$ $\quad 15 \times \frac{1}{4}$

Using Number Properties

Examples: Word problems for: $10 \times \frac{4}{7}$ $\quad 100 \times \frac{1}{5}$ $\quad 20 \times \frac{2}{9}$ $\quad 11 \times \frac{3}{4}$

Fractions Times Whole Numbers

I am learning how to multiply fractions by whole numbers.

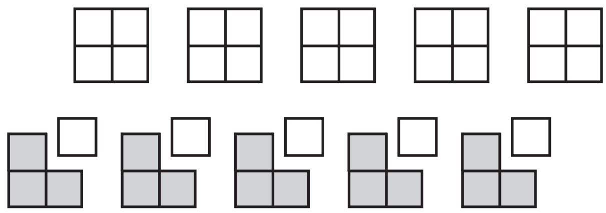
Equipment: Fraction materials.

Using Materials

Problem: "A caterer at a wedding supplies 5 pavlovas. At the end of the wedding, she sees 3 quarters of all the pavlovas have been eaten. How many pavlovas were eaten?"

Write $\frac{3}{4} \times 5$ on the board. Then students model wholes that can be cut into quarters:

Then they show $\frac{3}{4}$ of each pavlova being eaten:



Discuss why $\frac{15}{4}$ pavlovas were eaten.

Discuss why $\frac{3}{4} \times 5 = \frac{15}{4} = 3\frac{3}{4}$.

Examples: Word problems for: $\frac{3}{4} \times 2$ $\quad \frac{2}{5} \times 3$ $\quad \frac{5}{6} \times 3$ $\quad \frac{3}{4} \times 4$

Using Imaging

Do these by imaging. Fold back to building the fractions if needed.

Examples: Word stories for: $\frac{3}{4} \times 3$ $\quad \frac{2}{3} \times 3$ $\quad \frac{2}{3} \times 5$ $\quad \frac{3}{5} \times 4$

Using Number Properties

Problem: "Look at the answers to the examples above and work out how to find $\frac{3}{5} \times 7$ by using the numbers alone." *(Answer: $\frac{3}{5} \times 7 = \frac{21}{5} = 4\frac{1}{5}$.)*

Examples: Word stories for: $\frac{3}{10} \times 5$ $\quad \frac{8}{9} \times 7$ $\quad \frac{11}{12} \times 4$ $\quad \frac{3}{8} \times 16$

Problem: "Work out $\frac{10}{11} \times 3$ and $3 \times \frac{10}{11}$. What do you notice?"

(Answer: Reversing the order of multiplication does not alter the answer even though fractions are involved.)

Understanding Number Properties: What letters replace the question marks? $g \times \frac{h}{i} = \frac{? \times ?}{?}$ ***(Answer:*** $\frac{g \times h}{i}$***.)***

Understanding Number Properties: What letters replace the question marks? $\frac{h}{i} \times g = \frac{? \times ?}{?}$ ***(Answer:*** $\frac{h \times g}{i}$***.)***

EA
AA
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AP

A Fraction Times a Fraction

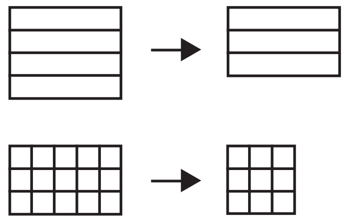
Multiplication of fractions follows easily from finding the area of a rectangle.

I am learning how to multiply fractions.

Equipment: Squared paper. Scissors.

Using Materials

Problem: "Maurice decides to plant $\frac{3}{4}$ of a rectangular field in carrots. Draw two copies of a 4 by 5 rectangle. Put one aside for later and cut out $\frac{3}{4}$ of the other one.



Next day Maurice realises he does not need all this $\frac{3}{4}$ area for carrots, so he decides to plant $\frac{3}{5}$ of this area in carrots. Draw vertical lines to create fifths and cut out $\frac{3}{5}$.

Compare the final cut-out shape with the spare copy of the original field. What fraction is in carrots?" *(Answer: 9 small squares are in carrots of the original 20 small squares. So the fraction is $\frac{9}{20}$.)*

Examples: Draw two 4 by 4 squares to work out $\frac{3}{4}$ of $\frac{3}{4}$.

Draw two 3 by 5 rectangles to work out $\frac{4}{5}$ of $\frac{2}{3}$.

Draw two 3 by 5 rectangles to work out $\frac{2}{3}$ of $\frac{4}{5}$.

Understanding Number Properties: Choose a size of rectangle suitable to work out $\frac{5}{6}$ of $\frac{4}{7}$ and then find the answer. ***(Answer: $\frac{20}{42}$ or $\frac{10}{21}$.)***

Using Number Properties

Problem: "Look at the answers to the examples above. What pattern do you notice?"

Examples: Work out: $\frac{7}{10} \times \frac{9}{10}$ $\quad \frac{3}{5} \times \frac{4}{5}$ $\quad \frac{8}{11} \times \frac{3}{10}$ $\quad \frac{6}{7} \times \frac{3}{100}$

Understanding Number Properties: What letters replace the question marks? $\frac{h}{i} \times \frac{j}{k} = \frac{? \times ?}{? \times ?}$ ***(Answer: $\frac{h \times j}{i \times k}$.)***

When Big Gets Smaller

I am learning that multiplying by a number less than 1 makes the answer smaller.

Equipment: Calculators. Worksheet (Material Master 8–15).

Using Number Properties

Problem: "Nara buys 0.345 kilograms of chocolates, which costs $9.78 per kilogram. Work out the approximate cost. Is the answer less than or more than $9.78?"

(Possible answer: 0.345 × 9.78 ≈ 3 tenths of 10 = 3. So the answer is less than $9.78.)

Use the calculator to find the cost of the chocolates and round sensibly.)

(Answer: 0.345 × 9.78 = 3.3741 = $3.37 to the nearest cent.)

Examples: Worksheet (Material Master 8–15).

Understanding Number Properties: How will you work out whether the answers to problems like 1.089 × 56.8 or 0.000123 × 567 is bigger or smaller than the numbers being operated on, 56.8 and 56.7?

Understanding Number Properties: z × y is always smaller than z when y is … ***(Answer: less than 1.)***
z × y is always bigger than z when y is … ***(Answer: greater than 1.)***

When Small Gets Bigger

Students' experience with whole numbers leads them to expect that multiplying results in an answer that is "larger" and that dividing results in an answer that is "smaller". However, 23.9 ÷ 0.0891 is *larger* than 23.9 not smaller. The whole number expectation holds for fractions greater than one but fails for fractions between one and zero.

EA
AA
AM
AP

I am improving my number sense by realising dividing by a positive number less than 1 makes the answer bigger.

Equipment: Paper strips. Scissors. Worksheet (Material Master 8–16).

Using Materials

Problem: "Maurice works at the supermarket. He has to cut up 4 kilograms of cheese into 0.5-kilogram packs for sale. How many packs will he make?" "What is $4 \div 0.5$?"

Write $4 \div 0.5$ on the board. Get the students to make a strip like this representing the 4 kilograms of cheese:

1 kg	1 kg	1 kg	1 kg

Then get them to cut the strip into "1-kilogram" pieces. Discuss how to create 0.5-kilogram packs. *(Answer: Cut each piece in half.)*

Do the cutting. How many pieces are there? *(Answer: 8.)*

Discuss why $4 \div 0.5 = 8$. "Is it a surprise that the answer is more than 4?"

Examples. Use paper and cutting for word stories: $3 \div 0.5$ $2 \div 0.25$ $3.5 \div 0.5$ $1 \div 0.25$ $2.5 \div 0.5$ ...

Using Imaging

Problem: "Morrie has 5 kilograms of chocolate. He puts the chocolate into separate packets of 0.25 kilograms. How many packets will Morrie make?" "What is $5 \div 0.25$?"

Write $5 \div 0.25$ on the board. Get the students to imagine what Morrie would do to 1 kilogram of chocolate. *(Answer: He would make 4 packets.)*

"How many bags can he make?" *(Answer: $5 \times 4 = 20$.)*

Write $5 \div 0.25 = 20$ on the board.

Examples: Imagine how to solve these problems. Word stories and problems for: $3.5 \div 0.5$ $1.25 \div 0.25$ $8 \div 0.5$ $2.5 \div 0.25$ ...

Using Number Properties

Examples: Worksheet (Material Master 8–16).

Understanding Number Properties: How will you decide whether the answers to problems like $345.67 \div 1.008$ or $2345.09 \div 0.012$ are bigger or smaller than the number being divided?

Understanding Number Properties: $z \div y$ is always smaller than z when y is ...
$z \div y$ is always bigger than z when y is ...

Estimation in Decimal Multiplication & Division Problems

I am using my number sense to check the answers in decimal fraction multiplication and division problems by estimation.

Equipment: Worksheet (Material Master 8–17).

Using Number Properties

Problem: "Estimate 0.503×58.7."

Discuss why $\frac{1}{2} \times 60 = 30$ is sensible.

Problem: "Martha gets $34.56 \div 0.98 = 33.8688$ on her calculator. Is she correct?"

(Answer: Martha must be wrong as division by a number less than 1 makes the answer bigger.)

Problem: "Estimate 59.89×0.091." *(Possible answer: $59.89 \times 0.091 \approx 60 \times 0.1$, and 1 tenth of 60 = 6.)*

Examples: Worksheet (Material Master 8–17).

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Calculating Percentage Changes

The need to calculate percentages is often not obvious to students. It really is a method of comparing fractions by giving both fractions a common denominator, namely hundredths. So it is useful to view percentages as hundredths.

I am learning to compare fractions by calculating percentages.

Equipment: Calculators. Worksheet (Material Master 8–18).

Using Number Properties

Problem: "Tom's Toothbrush Company makes a profit of $57,892.12 on sales of $456,789.23. A much bigger company United Industries makes a profit of $453,884.01 on sales of $4,489,000.89. Turn the fraction profit of each company into a percentage to see which company makes a better profit. Round the answers sensibly."

(Answer: Tom's fractional profit is $\frac{57892.12}{456789.23} = 0.126737051 \approx 0.127 = 12.7\%$. United Industries' fractional profit is $\frac{453884.01}{4489000.89} = 0.101110251 \approx 0.101 = 10.1\%$. So Tom's company makes a slightly better profit.)

Examples: Worksheet (Material Master 8–18).

Understanding Number Properties: Make up two fractions that are close to each other and use percentages to show which fraction is larger.

Estimating Percentages

As a check on percentage calculations, students need to know some standard simple fractions and their percentages equivalents. This will help estimating percentages as a check on calculation.

I am improving my number sense by learning to use a basic set of fractions equivalent to percentages to estimate percentages of given numbers.

Equipment: Worksheet (Material Master 8–19).

Preliminary

First check that the students know this table of simple fractions and percentage equivalents. These must be known off by heart.

10%	12.5%	20%	25%	50%
$\frac{1}{10}$	$\frac{1}{8}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{2}$

Using Number Properties

Problem: "Melissa works out 4.9% of $456 on a calculator and gets $22.34. Check Melissa's calculation by estimation."

Discuss why although $4.9\% \approx 5\% = \frac{1}{20}$, this is not useful in estimating.

(Answer: Finding $\frac{1}{20}$ of a number mentally is not easy.)

Discuss why starting at 10% is useful and then this amount is halved.

(Answer: 10% of 460 = $\frac{1}{10}$ of 460 = 46. Halving this to 5% is easy: $\frac{1}{2}$ of 46 = 23.)

Write 4.9% of $456 ≈ $23 on the board. Discuss whether Melissa's calculator answer is acceptable.

(Answer: Yes. 22.34 is near enough to 23.)

Problem: "16% of 3 961 TVs are found to be faulty at the factory and need repairs before they are sent for sale. About how many sets is this?"

(Possible answer: Discuss why 10% of 4 000 = 400, 5% of 4 000 = half of 400 = 200, so 16% of 3 961 ≈ 400 + 200 = 600.)

Examples: Worksheet (Material Master 8–19).

Understanding Number Properties: Explain how you would estimate 61% of a number. ***(Answer: 50% + 10%.)***

Percentages Problems in Two Steps

Students find mark-up problems easier to do by calculating the amount of mark-up as the first step, then adding this to the original as the second step. This delays the problem of percentages over 100%.

I am learning how to mark-up and mark-down in two steps and round answers sensibly.

Equipment: Calculators. Worksheet (Material Master 8–20).

Using Number Properties

Matt's Mark-up! 79%

Problem: "Matt buys shirts to sell in his clothing store.
He pays $24.85 for each shirt. What does he charge his customers?"

Discuss why the mark-up is 0.79×24.85 and write this on the board.

(Answer: 79% of 24.85 = 0.79 × 24.85 = 19.6315.)

Why should this answer not be rounded yet?

(Answer: As a general rule, calculations should be rounded after the last calculation to avoid errors.)

Discuss why the price to the customer is 19.6315 + 24.85 and write it on the board. Discuss how to enter this on a calculator.

(Answer: 19.6315 is already showing on the calculator so adding 24.85 does not require re-entering 19.6315.)

Discuss rounding the answer sensibly.

(Answer: 44.4815 = 44.48 to the nearest cent – fractions of a cent are not allowed.)

Problem: "At the vegetable shop, the owner Mrs Brown buys 450 tomatoes for resale. Before they can be sold, 14% of the tomatoes spoil. How many tomatoes does Mrs Brown sell?"

Discuss why 0.14×450 is the number of spoiled tomatoes and why Mrs Brown sells 387 tomatoes.

(Answer: 0.14 × 450 = 63, 450 – 63 = 387.)

Examples : Worksheet (Material Master 8–20).

Understanding Number Properties: Make up a mark-up or a discount problem and solve it. Explain the steps you took.

Percentage Increases & Decreases in One Step

Mark-ups in the one-step method are more efficient than the two-step method but are more difficult to understand. One significant obstacle to understanding is a common view that percentages over 100 are impossible. Yet one-step mark-ups require such percentages.

I am learning that percentages more than 100% are useful, and I am applying this to mark-up problems.

Equipment: Paper. Scissors. Calculators.

Worksheet (Material Master 8–21).

Using Materials

Problem: "The farmer begins the season with some cows. After they have had their calves the number of cows the farmer owns has increased by 50%."

Students cut a paper strip to represent 100%, and another strip to represent 50%. Discuss why the strips together represent 150% of the *original* number of cows.

100%	50%

Problem: "The population of a town drops by 6%. Cut a strip to show why the town's new population is 94% of the old population."

EA
AA
AM
AP

Using Number Properties

Problem: “The painter estimates the cost of painting Dale’s house will be $4,123 plus GST at 12.5%. Dale wants to know what he will have to pay altogether. He claims he can work out the answer by going 1.125 × 4 123. Is Dale correct? Explain.”

(Answer: Dale is correct. 100% + 12.5% = 112.5 % = 1.125.)

“What is the answer rounded sensibly?”

(Answer: 1.125 × 4 123 = 4 638.375 ≈ $4,638 to the nearest dollar.)

Examples: Worksheet (Material Master 8–21).

Understanding Number Properties: The Minister of Finance increases the rate of GST to 20%. Make up a problem for the cost of building a house when GST is added and solve the problem.

Squaring

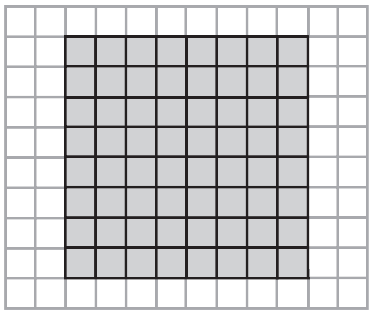
12^2 may be read as “12 to the power of 2” but normally it is read as “12 squared” because it represents the area of a square with sides 12. Areas of squares provide a useful geometrical introduction to squares and square roots.

I am learning that squaring a number gives the area of a square that has sides of that length.

Equipment: Squared paper. Square pieces of cardboard. Calculators.

Using Materials

Problem: “Matilda wants to make a square tile pattern on the floor. She draws a plan on squared paper. She wants to know how many small square tiles to buy without having to count them one by one.”



Get the students to draw 8 rows of 8 tiles on squared paper. Discuss how multiplication can help find the number of tiles needed.

(Answer: 8 × 8 = 64.)

Using Imaging

Problem: “Matilda plans another square that she describes as 10 rows by 10 tiles. Imagine how many tiles Matilda needs to build the square.” Fold back to drawing a picture on squared paper if needed.

(Answer: There are 10 rows each of 10 tiles and 10 × 10 = 100.)

Examples: By imaging, find the number of tiles needed to find the area of these squares:
4 by 4 6 by 6 7 by 7 9 by 9.

Using Number Properties

Problem: “Matilda plans another square that is 134 rows by 134 tiles. How many tiles will Matilda need to buy?”

(Answer: 134 × 134 = 17 956.)

Problem: “Matilda notices her calculator has a x^2 button. She investigates what it does.”

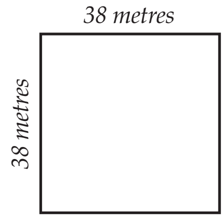
Get the student to input various values of x using the x^2 button and discuss what it does.

(Answer: x^2 is a shorthand for x times x.)

Problem: “Matilda uses the x^2 button to work out the number of tiles needed to build a square with dimensions 234 by 234. What is the answer?

(Answer: Press 234 x^2 = . The display shows 54756.)

Problem: "A contractor lays concrete slabs that are 1 metre by 1 metre. He has to pave a square that is 38 metres by 38 metres. Each slab costs $3. How much do the slabs cost altogether?



(Answer: There are 38^2 tiles needed = 1 444. So the cost is $3 \times 1\,444$ = $4,332.)

Examples: Find the area of these squares: 24 by 24 209 by 209 13 by 13 900 by 900 ...

Understanding Number Properties: The price of a tile is $$P$. What is the cost of a square pattern of d rows with d tiles in each row? ***(Answer: $d^2 \times \$P$.)***

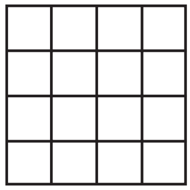
Square Roots

I am learning that finding the square root finds the length of the side given the area.

Equipment: Square cardboard pieces. Calculators.

Using Materials

Problem: "Zoë builds a large square from 16 small square tiles. How big is Zoë's square?"



Get the students to build the large square out of 16 tiles. Then discuss its dimensions.

(Answer: 4 rows of 4 tiles.)

Examples: Build large squares built with these numbers of small squares and give their size:
4 25 9

Using Imaging

Problem: "Zoë builds a large square from 36 small square tiles. Imagine how big the square is." Fold back to building the square if needed. *(Answer: 6 by 6.)*

Examples : Imagine large squares with these numbers of small squares and find their sizes:
49 100 81 ...

Using Number Properties

Problem: "Zoë builds a large square from 961 small square tiles. Describe the square Zoë builds with the aid of a calculator."

Discuss how the $\sqrt{x}$ or the $\sqrt{\ }$ button helps.

(Answer: $\sqrt{961} = 31$.)

Examples: Large squares are made with these numbers of small squares. Describe the large squares: 207 936 2 025 622 521 2 217 121 ...

Locating Square Roots

Most square roots do not come out nicely. For example $\sqrt{3}$ is an infinite non-recurring decimal starting 1.73205080 .... A method of finding square roots involves locating the answer between two numbers. In principal, this method will locate a square root to any desired accuracy.

I am learning how to find the square root of numbers without using a square root button on a calculator.

Equipment: Tables of squares (Material Master 8–22). Worksheet (Material Master 8–23).

Using Number Properties

Problem: "Meriana designs a *square* swimming pool, which she wants to have an area of 75 m^2. Unfortunately the square root button on her calculator is damaged, so she will have to find another way of finding square roots."

Stick these tables on the board from Material Master 8–22.

0	1	2	3	4	5	6	7	8	9	10
0	1	4	9	16	25	36	49	64	81	100

8	8.1	8.2	8.3	8.4	8.5	8.6	8.7	8.8	8.9	9
64	65.61	67.24	68.89	70.56	72.25	73.96	75.69	77.44	79.21	81

"How does the first table show the side of the pool is between 8 and 9 metres? How does the second table show the side of the pool is between 8.6 and 8.7 metres?"

8.6	8.61	8.62	8.63	8.64	8.65	8.66	8.67	8.68	8.69	8.7

Explore filling in the squares in this table to locate $\sqrt{75}$. *(Answer: It is between 8.66 and 8.67.)*

Examples: Worksheet (Material Master 8–23).

Understanding Number Properties: How could you find the square root to any desired accuracy of any number using only the square button on a calculator?

Cubes and Cube Roots

14^3 can be read as "14 to the power of 3", but usually it is read as "14 cubed" because it represent the volume of a cube with side 14. Volumes of cubes provides a useful geometrical introduction to cubes and cube roots.

I am learning that cubing a number produces the volume of a cube given the length of a side and finding the cube root finds the length of the side given the volume.

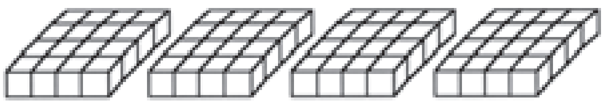
Equipment: Unilink cubes. Calculators.

Using Materials

Problem: "Norma builds a 4 by 4 by 4 cube layer by layer in linkable cubes. How many cubes does she need?"

Discuss why each layer is 4 by 4 cubes. Discuss how to describe this 4 by 4 by 4 cube in a written form. *(Answer: $4 \times 4 \times 4$.)*

"What is the alternative way of writing this?"
(Answer: 4^3.)



Using Imaging

Problem: "Norma builds cubes layer by layer in linkable cubes. Imagine how many cubes she needs for these bigger cubes." Fold back to building the cubes if needed. "How many cubes does Norma need to build these larger cubes: 3 by 3 by 3? 2 by 2 by 2? 5 by 5 by 5?"

Using Number Properties

Problem: "An ancient king decides to build a large cubic memorial to his reign from blocks of stone. Each stone is 1 metre by 1 metre by 1 metre.

He plans a memorial that is 89 metres by 89 metres by 89 metres. Use the y^x button on a calculator to find the number of blocks needed."

(Answer: 89 y^x 3 = 704 969.)

Examples: Use the y^x button on a calculator to work out: $34 \times 34 \times 34$ $45 \times 45 \times 45$ $0.89 \times 0.89 \times 0.89$ $67.2 \times 67.2 \times 67.2$ ...

Problem: "In a chemistry experiment, Lucy grows a blue crystal of copper sulphate that is cubic and that has a volume of 798 cubic millimetres. How long is each edge of the cube? Round sensibly."

(Answer: Using the $\sqrt[x]{y}$ button, Lucy enters 798 $\sqrt[x]{y}$ 3 = and the display shows 9.27543523. At best this is 9.3 mm. Note: on some calculators, the order is 3 $\sqrt[x]{y}$ 798 =.)

6 Minus 8 Does Work!

With problems such as 86 – 48, some students know how to solve 6 – 8. This leads to an interesting variation of the standard written forms for subtraction.

I am learning a method of subtracting using negative numbers.

Equipment: None.

Using Number Properties

Problem: "Hoana has $3 in the bank and withdraws $7. How much does she have in the bank?" Discuss how this can happen. *(Answer: Hoana owes the bank $4, so $3 - 7 = {}^-4$.)*

Hoana uses the method of subtracting using negative numbers to work out 73 – 27. She goes $3 - 7 = {}^-4$, then 70 – 20 = 50, then 50 – 4 = 46. So the method correctly shows 73 – 27 = 46."

$$\begin{array}{r} 73 \\ -\ 27 \\ \hline -\ 4 \\ 50 \\ \hline 46 \\ \hline \end{array}$$

Examples: Use Hoana's method to work out: 68 – 49 56 – 28 41 – 23 91 – 78 ...

Problem: "Use Hoana's method to work out: 423 – 157."

Examples: Use Hoana's method to work out: 345 – 189 4 520 – 5 170

Understanding Number Properties: Explain why Hoana's method does not involve negative numbers for every possible subtraction.

Finding Remainders

Division on a calculator does not provide remainders. Sometimes the remainder is obvious. For example, 3 457 ÷ 3 = 1152.333333 = $1152\frac{1}{3}$, so the remainder is 1. But 456 ÷ 37 = 12.324324; here the remainder is not obvious. Another method for finding the remainder is needed.

I am learning to find remainders using a calculator.

Equipment: Calculators.

Using Number Properties

Problem: "Joy works out 39 803 ÷ 3 on her calculator and gets 13267.6666. Joy claims the remainder is 2. Is she correct?

(Possible answer: Yes. 0.6666666 derives from 2 ÷ 3 so the remainder is 2.)

Examples: Find the remainders for these divisions: 10 334 ÷ 3 1 347 ÷ 4 233 ÷ 5 12 789 ÷ 2 124 ÷ 3

Problem: "Joy has to pack 5 673 sweets into packets of 34 sweets. How many loose sweets will be left?"

Do 5673 ÷ 34 on a calculator. *(Answer: 166.8529412.)*

Discuss why this means there are 166 packets, which use 166 × 34 = 5 644 sweets. So there are 5 673 – 5 644 = 29 sweets left over.

Examples: Find the remainders for these divisions: 10 334 ÷ 27 11 327 ÷ 67 123 833 ÷ 27 62 789 ÷ 967 345 810 ÷ 1 614

Understanding Number Properties: For any whole numbers d and b, how do you find the remainder for $d \div b$? ***(Answer: Work out d ÷ b on a calculator. Ignore the decimal part to get the quotient. The remainder is d minus the quotient times b.)***

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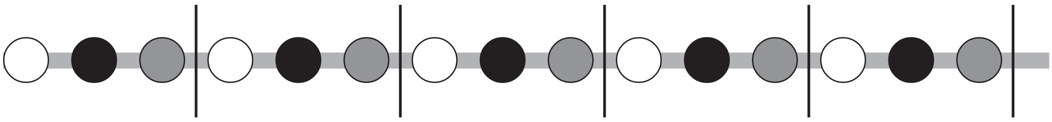
Applying Remainders

I am learning to use remainders to solve patterning problems.

Equipment: Counters in at least 3 colours.

Using Number Properties

Problem: "Michael builds a string of beads using 3 colours, starting at white, black, then red then repeating the pattern.



He continues building using 103 beads altogether. What colour is the last bead?"

(Possible answer: 103 ÷ 3 = 34 remainder 1. So there are 34 groups of 3 and the last one is white.)

Discuss what colour the last one will be if the remainder is 1, 2, or 0.

Examples: Find the colour of the last bead if there are these numbers of beads: 56 123 678 828 8 987

Understanding Number Properties: Make up a bead problem like Michael's where the number of beads is over 10 000.

Problem: "Michael builds a pattern of counters with four colours A, B, C, and D. So the counters look like ABCD ABCD ABCD ABCD .... What colour is the 45 679^{th} bead?"

(Possible answer: By calculator: 45 679 ÷ 4 =11419.75. The last bead will be C.)

Example: Find the colour of the last bead if there are these numbers of beads: 5 489 12 788 823 451 34 500 28 987

Problem: "Starting at 12 o'clock, Michael records the hour hand of a 12-hour clock going around for 12 345 hours. What time does the clock show at the end of the 12 345 hours?"

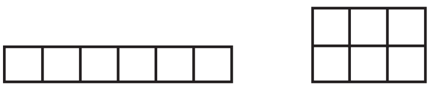
Prime Numbers

I am learning that if I represent numbers by tiles arranged in rectangles, prime numbers can only be arranged one way and other numbers have more than one way.

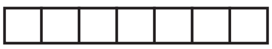
Equipment: Square tiles or cards.

Using Materials

Problem: "Gerry builds rectangles out of 6 square tiles. Find all the ways of building rectangles with these tiles."

(Answer:  *.)*

"Repeat for 7 tiles."

(Answer:  *.)*

Discuss why 7 is called a prime number and 6 is not.

(Answer: There is only one way of building 7, namely 7 tiles in a single row. Whereas 6 can also be built as 2 rows of 3.)

Examples: By building rectangles out of tiles determine which of these numbers are prime and which are not: 5 8 11 9 15 13 16

Understanding Number Properties: Explain why 2 is the only even prime number.

Using Number Properties

Problem: "7 has factors 1 and 7, and 7 is a prime number.

6 has factors 1, 2, 3, and 6, and 6 is *not* a prime number.

Complete this general rule:

If we look at all the factors of a number, we know it is a prime number if ..."

Divisibility Tests

Important work involving finding factors of numbers such as Lowest Common Multiples (LCMs), Highest Common Factors (HCFs), and the addition or subtraction of fractions is simplified if students have some tests to spot simple factors like 2, 3, 4, 5, 6, and 10.

I am learning some divisibility tests.

Equipment: Worksheet (Material Master 8–24).

Using Number Properties

Problem: "Prakesh believes that 2 is a factor of a number only when the last digit is even. Explore this claim. Is Prakesh correct?" *(Answer: Yes.)*

Problem: "Prakesh writes down the multiples of 5 in order: 5, 10, 15, 20, 25, ... He spots a way of telling whether a number has 5 as a factor or not. What does he spot?"

(Answer: All multiples of 5 end in 0 or 5.)

Problem: "To find whether 4 is a factor of 34 984, Prakesh has an inspiration. He splits the number into 34 900 and 84. He is certain 34 900 has 4 as a factor. Why?"

(Answer: $34\,900 = 349 \times 100$ and $100 \div 4 = 25$. So 34 900 has a factor 4.)

"84 has 4 as a factor because $84 \div 4$ has no remainder. So explain why 34 984 has 4 as a factor."

Examples: Which of these numbers are multiples of 4? 345 638 232 12 002 1 295 904 180 008 ...

Problem: "Explore this claim: 9 is a factor only if the digit sum is divisible by 9."

(Answer: It is correct. This is hard to show. An example:

$$\begin{aligned} 2\,745 &= 2 \times 1\,000 + 7 \times 100 + 4 \times 10 + 5 \\ &= 2 \times 999 + 7 \times 99 + 4 \times 9 + (2 + 7 + 4 + 5) \\ &= 9 \times (2 \times 111 + 7 \times 11 + 4) + (2 + 7 + 4 + 5) \end{aligned}$$

9 is a factor of $9 \times (2 \times 111 + 7 \times 11 + 4)$. So 9 is a factor of 2 745 only if 9 is a factor of the sum of the digits $(2 + 7 + 4 + 5)$, which it is.)

Similarly 3 is a factor of a number only when the digit sum is divisible by 3.

Examples: Worksheet (Material Master 8–24).

Understanding Number Properties: Explain how you would test whether 36 is a factor of a number.

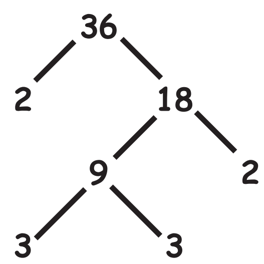
Factor Trees

I am learning to use factor trees to produce all the prime factors of a number.

Equipment: None.

Using Number Properties

Problem: "Barry wants to factorise 36. He notes the following:
$36 = 2 \times 18$, $36 = 4 \times 9$, $36 = 6 \times 6$. He draws a complete factor tree for $36 = 2 \times 18$ as shown. Draw factor trees that start at $36 = 4 \times 9$, and $36 = 6 \times 6$." Discuss why the ends of the trees *all* show $36 = 2 \times 3 \times 3 \times 2$.



Examples: Noting $24 = 2 \times 12$, $24 = 8 \times 3$, $24 = 6 \times 4$, draw factor trees that *all* show $24 = 2 \times 2 \times 2 \times 3$ at the end.

Noting $30 = 3 \times 10$, $30 = 6 \times 5$, $30 = 2 \times 15$, draw factor trees that *all* show $30 = 2 \times 3 \times 5$ at the end.

Problem: "Julia wants to factorise 144. She decides to use divisibility rules. 2 is a factor of 144, so $144 = 2 \times 72$. 2 is a factor of 72, so $72 = 2 \times 36$. Julia continues testing divisibility by 2 until she arrives at $18 = 2 \times 9$. She notes $9 = 3 \times 3$, and so she stops. Follow Julia's steps, which show $144 = 3^2 \times 2^4$. Use a factor tree if this helps."

Examples: Reduce these numbers to their prime factors. Use the power notation for the answers:
180 120 1 000

Understanding Number Properties: Choose a number of your own over 150 and reduce it to its prime factors.

Adding Sequences

The famous mathematician Carl Friedrich Gauss was asked as a five year old to work out $1 + 2 + 3 + ... + 99 + 100$. He almost instantly replied the answer was 5050. How could he have got the answer so fast?

I am learning how to add up some series of numbers quickly.

Equipment: Two sets of cards numbered 1 to 20.

Using Materials

This is the method Gauss used to add up the numbers 1 to 10.

Place cards from 1 to 10 in order

1	2	3	4	5	6	7	8	9	10

Place 10 under 1, 2 under 9, and so on.

1	2	3	4	5
10	9	8	7	6

Discuss the total of each pair. *(Answer: it is 11.)*

Discuss why the total is $5 \times 11 = 55$. *(Answer: There are 5 pairs each adding to 11.)*

Repeat for $1 + 2 + 3 + 4 + ... + 8$, $1 + 2 + 3 + 4 + ... + 12$, $1 + 2 + 3 + 4 + ... + 14$

Using Number Properties

Solve Gauss' problem: $1 + 2 + 3 + 4 + ... + 98 + 99 + 100$.

(Answer: 1 and 100 = 101, 2 and 99 = 101 ... So the series adds up to $50 \times 101 = 5050$.)

Repeat for $1 + 2 + 3 + 4 + ... + 999 + 1000$, $1 + 2 + 3 + 4 + ... + 46$, $1 + 2 + 3 + 4 + ... + 88$.

Understanding Number Properties: Find a formula for $1 + 2 + 3 + 4 + ... + 2n$. ***(Answer: $n(2n + 1)$.)***

The Sieve of Eratosthenes

Eratosthenes, who lived in Greece from about 276 to 195 BC, invented a system to find prime numbers. It consists of crossing out every second number except 2 on a grid then every third number except 3, and so on. The numbers not crossed out are prime. (A discussion about 1 being a special number in that it is neither prime nor non-prime may be worthwhile.)

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I am learning how the Sieve of Eratosthenes produces prime numbers.

Equipment: A grid of numbers 1 to 200 (Material Master 8–13).

Using Materials

Give each student a copy of the grid. Notice the number 1 is shaded to indicate it is special.
2 is prime and ringed. Discuss how to cross out all the multiples of 2. *(Answer: For example cross out whole columns at a time.)*

Ring 3 and cross out all multiples of 3 (that is 6, 9, 12 ...). Ring 5 and cross out multiples of 5. Continue until all the prime numbers ≤ 200 are ringed.

Discussion: Why is this called a sieve?

Understanding Number Properties: Describe how you would determine whether 349 is prime using the Sieve of Eratosthenes. Don't actually do it. Is the method generally useful? Consider for example, testing whether 179 781 is prime.

Leap Years

I am learning to work out which years are leap years.

Equipment: None

Using Number Properties

Leap years arise because the true length of a solar year is almost $365\frac{1}{4}$ days. Discuss why there is the need for an extra day every 4 years. *(Answer: Every 4 years there are 4 quarter days extra, that is to say 1 day extra.)*

The Olympic games are held every leap year. The most recent leap years were 2004, 2000, and 1996. How can we determine which years the Olympic Games will be held?

Simplify the problem by telling the students that the first leap years were 4, 8, 12, 16 AD. Discuss the pattern. *(Answer: Leap years are all divisible by 4.)*

Is the year 3454 going to be a leap year? *(Answer: Using the divisibility by 4 rule 54 ÷ 4 has a remainder so 3454 is not an Olympic year.)*

Examples: Which of these were leap years? 1978 1943 1876 1874 1767 1604

Question: The Olympic Games were not held in 1916, 1940 and 1944. These were leap years. What went wrong? *(Answer: These years were in World War 1 and World War 2, so the Games were cancelled.)*

Turn of the Century

The length of a solar year is not quite exactly $365\frac{1}{4}$ days. This is too long by 11 minutes 14 seconds per year. So every 128 years the calendar has gained almost exactly 1 day.

I am learning which turn of the century years are leap years.

Equipment: Calculators.

Using Number Properties

Discuss why an error 11 minutes 14 seconds per year amounts to an error of almost exactly 1 day every 128 years. *(Answer: 11 minutes 14 seconds = $11\frac{14}{60}$ minutes = 11.2333333 minutes. So in 128 years there is a gain of 128 × 11.2333333 minutes = 128 × 11.2333333 ÷ 60 hours = 128 × 11.2333333 ÷ 60 ÷ 24 days = 0.9985 days which is almost exactly 1 day.)*

Discuss why using $365\frac{1}{4}$ days as the length of a year every 400 year period is very nearly 3 days too long. *(Answer: 3 × 128 = 384 ≈ 400.)*

EA
AA
AM
AP

Pope Gregory declared 1600 would be a leap year but 1700, 1800, 1900 would not. This pattern continues. 2000 was a leap year but 2100, 2200, and 2300 will not be.

Discuss why 3200 will be a leap year but 3400 will not be.

Examples: Which of these years will be a leap year? 3300 4600 3000 4400

Note: Another small correction is not due until 4909 which will concern no-one.

Understanding Number Properties: Find a test to determine whether a turn of century year is a leap year or not. ***(Answer: Determine the number of centuries in the year, eg 4600 = 46 centuries. Test whether the number of centuries is divisible by 4. If it is then the year is a leap year. Alternatively divide the year by 400. If there is no remainder the year is a leap year.)***

The Lunar Year

Many ancient cultures based their calendars on a year that consisted of 12 lunar months. This led to an error of about 9 days a year which had to be corrected.

I am learning about lunar based calendars.

Equipment: Calculators.

Using Number Properties

Discuss how to work out the length of 12 lunar months.

(Answer: $12 \times 29.5306 = 354.3672$ days.)

Find the difference between a lunar year and a true year.

If a lunar calendar is not corrected, how long will it take for a date in the middle of summer to drift to become a date in the middle of winter?

Some Calendar Facts	
1 lunar month	29.5306 days
1 true year	365.24199 days

(Answer: Half a year is about 180 days. Each year the drift between the lunar year and the true year is about 9 days. So, if no correction is made, after about 20 lunar years a mid-summer date becomes a mid- winter date.)

Learning Experiences for Advanced Proportional Part-Whole

Activities from Other Books

Hot Shots
Page 31, Book 7. (2007 edition)
I am learning to work out percentages of amounts.

Folding Fractions and Decimals
Page 36, Book 7.
I am learning to multiply fractions and decimals.

Mixing Colours
Page 34, Book 7.
I am learning to compare ratios and proportions.

New Activities

Multiplication of Decimal Fractions

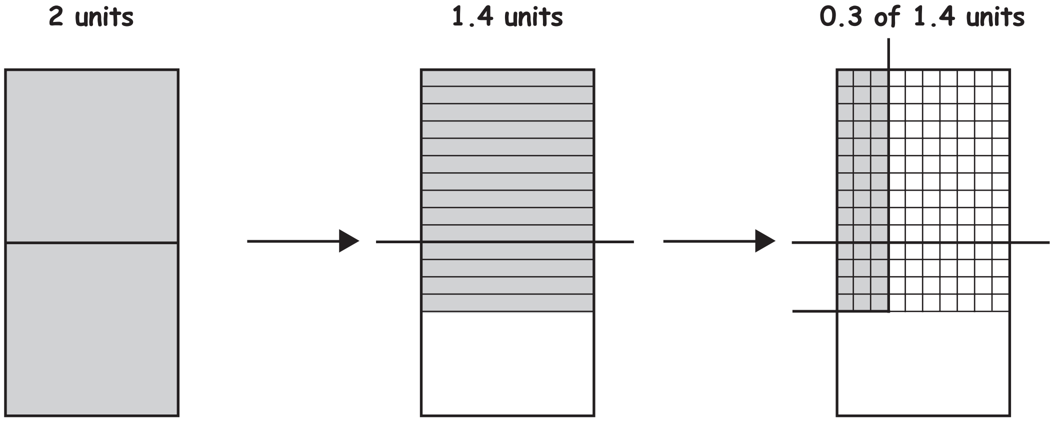
I am learning how to multiply decimal fractions by finding areas of rectangles.

Equipment: 10 by 10 grids (Material Master 8–34).

Using Material

Problem: "John wants to find the area of a rectangle 1.4 by 0.3. Because 1 tenth × 1 tenth = 1 hundredth, John decides to make one unit of area equal the square with 100 little squares."

Get the students to use Material Master 8–34 to draw the following:



Discuss how the pictures show 0.3 of 1.4 = 42 little squares. *(Answer: 3 columns of 14 are shaded in the third picture which gives 3 × 14 = 42 little squares.)*

Discuss what fraction of a square unit is shaded. *(Answer: $\frac{42}{100}$ = 0.42.)*

Discuss the quick way of seeing 0.3 × 1.4 = 0.42. *(Answer: 3 tenths of 1.4 is less than 1, in fact about 0.5. The answer has 42 in it. So it must be 0.42.)*

Examples: Repeat for: 1.2 × 0.2   1.2 × 1.2   0.9 × 0.8   1.2 × 1.1

Using Number Properties

Problem: "John notices a pattern in the answers above. He immediately reasons the answer to 2.5 × 1.3 must be 3.25. Explain."

(Answer: John calculates that, ignoring the decimal point, 25 × 13 = 325. And 2.5 × 1.3 is between 2 × 1 = 2 and 3 × 2 = 6, so the answer must lie between 2 and 6, which means that the point must follow the 3.)

1.2 × 0.2 = 0.24
1.2 × 1.2 = 1.44
0.9 × 0.8 = 0.72
1.2 × 1.1 = 1.32

EA
AA
AM
AP

Problem: "John's calculator is malfunctioning. It gives digits in the answer correctly but is not showing the decimal point. He puts 178 ÷ 0.0892 into the calculator and gets 1995515695 on the screen. Where should the decimal point go?"

(Answer: 178 ÷ 0.0892 ≈ 200 ÷ 0.1 = 2 000. So the answer is 1995.515695.)

Examples: Estimate the answer then put the decimal point in the correct place:
9.789 × 0.12 = 117468 10.11 × 9.9 = 100089 4.9 × 0.51 = 2499 78.6 ÷ 0.88 = 893181818
6.76 ÷ 0.026 = 26 2.5 × 2.5 = 625 ...

Understanding Number Properties: Describe how you would know where the decimal point will be in the answers to a) 234.567 ÷ 24.987 b) 0.0123 × 678.34

Recurring and Terminating Decimal Fractions

On a calculator, 1 ÷ 9 = 0.111111 ..., yet 456 ÷ 62 500 = 0.007296. This activity uses connections between a number and its divisor that allow prediction as to whether the quotient terminates or recurs.

I am learning which divisions have a terminating decimal answer and which divisions have a recurring decimal answer.

Equipment: Calculators.

Using Number Properties

Problem: "Write down the prime factorisations of the numbers 8 16 20 25 32 50 64 100 125

What do you notice?" *(Answer: The only prime factors are 2 and/or 5.)*

Problem: "These problems use divisors with prime factors of 2 and/or 5, if you consider them as whole numbers, e.g. seeing 12.5 as 125. Find the answer on a calculator:
231 ÷ 20 29 ÷ 1.6 89 ÷ 0.25 14 ÷ 12.5 73 ÷ 3.2 45 ÷ 8

What do you notice about all the answers?" *(Answer: They all terminate because 2 and 5 divide equally into ten.)*

Problem: "Hiria works out 47 ÷ 6 and gets 7.8333333. Discuss what the actual answer is."

(Answer: It is 7.8333333333 ..., where 3 repeats forever.)

Write 7.8333333333 ... as $7.8\dot{3}$ on the board.

Problem: "Solve these problems on a calculator and write the answers as recurring or terminating decimals:
14 ÷ 3 28 ÷ 1.2 4.8 ÷ 0.9 15 ÷ 11 2.6 ÷ 15 3 ÷ 0.7 27 ÷ 6 4.9 ÷ 1.4
39 ÷ 15 3.8 ÷ 15 26 ÷ 1.8 27 ÷ 0.18

Problem: "What do you notice about the prime factors of these divisors, considering them as whole numbers? How can we tell from that whether the decimal will terminate or recur?"

(Answer: All of the divisors have prime factors other than 2 and 5, e.g. 12 = 2 × 2 × ***3****. These other prime factors do not divide equally into ten. If any of these other primes do not divide equally into the starting number, the answer will be a recurring decimal.)*

Examples: Predict without using a calculator which of these divisions will have a recurring decimal answer and which answers will terminate:
1.7 ÷ 64 32 ÷ 0.9 2.55 ÷ 1.2 4.2 ÷ 11 1.11 ÷ 6 28.7 ÷ 64

Understanding Number Properties: The multiplicative relationship between the number being divided and the prime factors of the divisor determines whether a decimal answer will recur or terminate. If the prime factors are only 2 and/or 5 the answer will always terminate, e.g. 28.7 ÷ 64 =0.4484375. If other primes are involved the answer will only terminate if the number is divisible by each of these prime factors, e.g. 1.11 ÷ 6 = 0.185, as 111 is divisible by three. Otherwise the answer will be a recurring decimal.

Extension Activity

Problem: "Find 34 ÷ 7 and explain why the answer is a recurring decimal whose repeating cycle cannot be more than 6 digits." *(Answer: 34 ÷ 7 = 4.857142857... This indicates 34 ÷ 7 =* $4.\overline{857142}$*. The cycle is 6 digits long. It cannot be more than 6 digits because a number ÷ 7 has 7 possible remainders 0, 1, 2, 3, 4, 5, 6 at any stage of the long division. If the remainder is over 0, the division ceases, so the answer in such a case is not a recurring decimal. So, if the decimal is recurring there are only 6 possible remainders, which must be reused at most after 6 applications of the division process.)*

Writing Very Large Numbers

Most students are intrigued by large numbers. Facts like the amount of money earned last year throughout the world by 5 billion people being about 20 trillion dollars are intriguing. Numbers of this size are hard to read, so a standard form using powers of 10 is preferable.

I am learning how to write very large numbers in a compact standard form and am learning about the addition of powers rule.

Equipment: None.

Using Number Properties

Problem: "What does 10 000 × 100 000 equal? Write it in power notation."
"Explain why $10^4 \times 10^5 = 10^9$."

Problem: "How would you work out 1 000 000 × 100 000 000 using powers?"

(Answer: 1 000 000 × 100 000 000 = $10^6 \times 10^8 = 10^{14}$*.)*

Problem: "In a programme on TV, a scientist claimed there are 10 billion trillion stars in the universe. Write this amount with zeros and in the power form."

(Answer: 10 billion trillion = 10 000 000 000 × 1 000 000 000 000 = $10^{10} \times 10^{12} = 10^{22}$*.)*
Note: The American system, where 1 billion = 1 thousand million and 1 trillion = 1 thousand billion is used here. The British system, where 1 billion = 1 million million and 1 trillion = 1 million billion, is not used since it breaks the pattern where the names hundreds, tens, and ones are repeated every 3 digits.)

Problem: "A googol is a 1 followed by 100 zeros. Write 1 billion × 10 googols in power form."

(Answer: 1 billion × 10 googols = $10^9 \times 10^{101} = 10^{110}$*.)*

Understanding Number Properties: Complete: = $10^m \times 10^n = 10^{\boxed{}}$ ***(Answer:*** 10^{m+n}***.)***

Highest Common Factors (HCFs)

Highest Common Factors (HCFs) are needed for a range of problems involving fractions and eventually algebra. They require good understanding of factors and the skill of instant recall of the basic multiplication facts (tables). Failure to know tables effectively shuts students out of working with factors and therefore highest common factor work.

I am learning to spot common factors in numbers and so find the Highest Common Factor (HCF).

Equipment: None.

Using Number Properties

Problem: "Brian, Peter, and Murray buy packets of cakes. Brian has 12 cakes, Peter 15 cakes, and Murray 9 cakes. How many cakes are there in each packet? (Assume there is more than 1 in a packet.)"

Discuss why 12, 15, and 9 are all multiples of the number of cakes in a packet. So the number in a packet must be 3.

EA
AA
AM
AP

Problem: "This time Brian, Peter, and Murray buy packets of lollies. Brian has 60 lollies, Peter 40 lollies, and Murray 48 lollies. Write down the factors of 60, 40, and 48 and find the common factors."

(Answer: 60 has factors 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60.
40 has factors 1, 2, 4, 5, 8, 10, 20, 40.
48 has factors 1, 2, 3, 4, 6, 8, 12, 16, 24, 48.
The common factors in all lists are 2 and 4 – ignore 1.)

What is the biggest number of lollies that could be in each packet? *(Answer: 4.)*

Examples: Find the highest common factor for these sets of numbers. Draw factor trees only if you have to: 54, 36, 24   36, 72, 24   50, 80, 45   36, 42, 6 000   500, 15, 25   21, 54, 18 ...

Problem: "Maureen, Chen, and Anna open packets of lollies to put in bowls for a party. Maureen's bowl has 68 lollies, Chen's bowl has 51 lollies, and Anna's bowl has 34. How many lollies were in each packet?"

(Possible answer: Using prime factors $68 = 2 \times 34 = 2 \times 2 \times 17$, $51 = 3 \times 17$, $34 = 2 \times 17$. *So the HCF is 17. So one packet has 17 sweets.)*

Problem: "Norrie wants to find the highest common factor of 372 and 558. He decides to write each number as a product of prime numbers. Find the HCF of 372 and 558."

(Answer: By using divisibility tests $372 = 2 \times 186 = 2 \times 2 \times 93 = 2 \times 2 \times 3 \times 31$.
And $558 = 2 \times 279 = 2 \times 3 \times 93 = 2 \times 3 \times 3 \times 31$. *2 and 31 are common factors so is* $2 \times 31 = 62$.
So 62 is the HCF.)

Examples: Find the HCFs of these sets of numbers: 195 and 312   312 and 408   162 and 522   208, 130, and 234   378, 224, and 140 ...

Lowest Common Multiples (LCMs)

I am learning to spot common multiples in numbers and so find the Lowest Common Multiple (LCM).

Equipment: Calculators.

Using Number Properties

Problem: "Two salesmen are in Timaru today. John returns to Timaru every 4 days, and Martin visits every 6 days. What days will they both be in Timaru?"

Draw the table on the board and discuss how to fill it in. The first few entries have been made.)

Days

John	4	8	12	16						
Martin	6	12	18							

"What are the common multiples of 4 and 6? What is the LCM of 4 and 6?"

(Answer: 12, 24, 36, ... LCM = 12.)

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AP

Problem: "John visits Greymouth every 20 days, and Martin visits every 12 days. How often are they in Greymouth together?"

Examples: Find the LCMs of these pairs: 6 and 9 12 and 18 20 and 25 25 and 10 5 and 6 10 and 11

Problem: "Josie knows that listing common multiples to find the LCM of 51 and 136 will be tedious. So she summarises the answers above and looks for a pattern. Josie spots a link between the LCM, the product of the numbers, and the HCF. What is it? Explore whether the pattern always works."

Numbers		LCM
4	6	12
20	12	60
6	9	18
12	18	36
20	25	100
25	10	50

(Answer: LCM = product ÷ HCF in all cases.)

Examples: Find the LCMs of these pairs of numbers: 49, 112 156, 65 738, 615 910, 728 459, 1 917 ...

Understanding Number Properties: The LCM of any two numbers *a* and *b* can be found by ... ***(Answer: dividing a × b by the HCF of a and b).***

Comparing Chalk and Cheese

Adding and subtracting seem easy, but care must be taken that the units involved are the same. If they are not the same, they must be converted. This becomes the basic notion needed in the next activity concerning adding and subtracting fractions.

I am learning to convert unlike units in order to add or subtract them.

Equipment: None.

Using Number Properties

Problem: "Add $4.34 and 108 cents in two ways."

(Answer: Convert the cents into dollars and cents so $4.34 + $1.08 = $5.42. Convert dollars and cents into cents, so 434 cents + 108 cents = 542 cents.)

Problem: "Find 109 cm – 80 mm."

(Answer: 1 090 mm – 80 mm = 1 010 mm or 109 cm – 8 cm = 101 cm.)

Examples: Work these problems out in two different ways: 100 mm + 1.5 m 5.6 kg – 700 g 2 min 30 sec + 40 sec 2 hrs 30 min + 50 min

Example: $US1 = $NZ2. Fiona has $US10 and $NZ30 in her purse. How much money does she have altogether in either currency?

Understanding Number Properties: Make up an addition or subtraction problem with a mixture of millilitres and litres and solve the problem in two ways.

Adding and Subtracting Fractions

The fact that many students struggle to understand how to add and subtract fractions explains why the topic has been pushed back to level 5 of the mathematics and statistics learning area of *The New Zealand Curriculum.* The basic notion required is that when fractions have different denominators, they must be renamed to have a common denominator.

I am learning that to add and subtract fractions efficiently I need to first find the Lowest Common Multiple and then proceed with the problem.

Equipment: Worksheet (Material Master 8–26).

EA
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AP

Using Number Properties

Problem: "Mele wants to find $\frac{1}{2} + \frac{1}{6}$. Why can't the fractions be added directly?"

(Answer: Halves and sixths are unlike.)

"How will Mele proceed?"

(Answer: She will convert $\frac{1}{2}$ to $\frac{3}{6}$, so $\frac{1}{2} + \frac{1}{6} = \frac{3}{6} + \frac{1}{6} = \frac{4}{6}$.)

Examples: Work out $\frac{1}{2} - \frac{1}{6}$ $\quad \frac{1}{8} + \frac{1}{2}$ $\quad \frac{7}{8} - \frac{1}{2}$ $\quad \frac{11}{12} - \frac{5}{6}$ $\quad \frac{2}{3} + \frac{5}{12}$ ...

Problem: "Mele wants to work out $\frac{8}{9} - \frac{5}{6}$. How will Mele get around the problem of unlike denominators?" *(Answer: Mele must find a way to convert both fractions to have a common name.)*

List the equivalent names for $\frac{8}{9}$ and $\frac{5}{6}$ on the board.

(Answer: $\frac{8}{9} = \frac{16}{18} = \frac{24}{27} = \frac{32}{36} = \frac{40}{45} = \frac{48}{54}$ ... $\quad \frac{5}{6} = \frac{10}{12} = \frac{15}{18} = \frac{20}{24} = \frac{25}{30} = \frac{30}{36} = \frac{35}{42} = \frac{40}{48} = \frac{45}{54}$ ...)

Discuss which denominators appear in both lists. *(Answer: 18, 36, 54 ...)*

Discuss why $\frac{8}{9} = \frac{16}{18}$ and $\frac{5}{6} = \frac{15}{18}$ are the best fractions to use to rename $\frac{8}{9}$ and $\frac{5}{6}$ and why $\frac{8}{9} - \frac{5}{6} = \frac{1}{18}$.

Examples : Worksheet (Material Master 8–26).

Understanding Number Properties: Make up an addition problem for fractions with unlike denominators and then solve it.

Ratios with Whole Numbers

Students, and indeed some textbook writers, often confuse ratios and fractions because of the similarity of their equivalence laws. For example, 3 : 4 = 12 : 16 is speciously the same as $\frac{3}{4} = \frac{12}{16}$. However, a ratio of 3 blue to 4 yellow, in fact, means $\frac{3}{7}$ of the mix is blue and $\frac{4}{7}$ is yellow. To avoid this confusion, ratios are introduced to students as 3 or more quantities rather than 2.

I am learning to use equivalent ratios.

Equipment. Worksheet (Material Master 8–27).

3 cups of flour
1 cup of sugar
2 eggs
100 g almonds
150 g butter
1 tspn of vanilla

Using Number Properties

Problem: "Georgie sees a cake recipe in a book. She writes the quantities as the ratio 3 : 1 : 2 : 100 : 150 : 1. Georgie wants to make 3 cakes. Write the quantities needed as a ratio."

(Answer: Everything is tripled so 9 : 3 : 6 : 300 : 450 : 3 is the recipe for 3 cakes.)

Problem: "Walter plans to make a very large cake with the recipe. He plans to use the ratio 30 : 10 : 20 : 1 000 : 500 : 10. Walter has made a mistake in his calculations? What is the mistake?"

(Answer: Walter should be multiplying all quantities by 10. So for butter 10 × 150 = 1 500 not 500.)

Problem: "Three university students paint a house in their holidays to earn money. They don't all work the same hours. Jude works for 40 hours. Look at the pay slip and decide how long Melanie and Aaron worked for."

Pay	
Melanie	**$400**
Jude	**$800**
Aaron	**$600**

Discuss how to reduce 400 : 800 : 600 to its lowest equivalent ratio.

(Answer: Link this to the HCF of 400, 800, 600 being 200. So 400 : 800 : 600 = 2 : 4 : 3 by dividing all numbers by 200. So 2 : 4 : 3 = 20 : 40 : 30 by multiplying through by 10. So Melanie has worked for 20 hours and Aaron has worked for 30 hours.)

Examples: Worksheet (Material Master 8–27).

Understanding Number Properties: Are the ratios $3a : 3b : 3c$ and $a : b : c$ the same? Explain. ***(Answer: Yes because dividing each of 3a:3b:3c by 3 gives a:b:c.)***

Comparing by Finding Rates

I am learning to compare different things by calculating rates.

Equipment: Calculators. Worksheet (Material Master 8–28).

Problem: "Julie drives 210 kilometres in 3 hours, and Marie drives 340 kilometres in 5 hours. Who is driving faster on average?"

Discuss how far Julie and Marie travel in 1 hour.

(Answer: 210 ÷ 3 = 70 km/h, 340 ÷ 5 = 68 km/h. So Julie is travelling faster.)

Problem: "A 500 gram packet of Wheatie Flakes costs $1.85, and a 800 gram packet of Wheatie Flakes costs $2.88. The supermarket shows the costs in cents per 100 grams. What do the labels show? Which packet is better value?"

(Answer: 185 cents ÷ 5 = 37 cents per 100 grams. 288 cents ÷ 8 = 36 cents per 100 grams. So the bigger packet is better value.)

Problem: "Gather data about different sized packets of a product from a supermarket. Compare their value by finding the cost in cents per 100 grams for each of the sizes."

Examples: Worksheet (Material Master 8–28).

Understanding Number Properties: k grams of flour costs \$$j$. How would you work out the cost of flour per 100 grams? ***(Answer: \$$\frac{100j}{k}$ per 100 grams.)***

Sharing in Ratios

I am learning to share quantities in a given ratio.

Equipment: Worksheet (Material Master 8–29).

Using Number Properties

Problem: "Mrs Norris hires students to work on her garden. She agrees to pay them $400 total. Diane, Sonya, and Mere work 13 hours, 10 hours, and 17 hours respectively on the garden. How much pay will Sonya receive?"

Discuss why the ratio of hours 13 : 10 : 17 leads to Sonya receiving $100.

(Answer: 13 + 10 + 17 = 40. $\frac{10}{40} \times 400 = 100$.)

Problem: "Two friends buy a house that costs $193,500. Bernie contributes $89,500, and Bronwyn contributes $104,000. Five years later, they sell it for $234,000. How much money should each of them receive?

(Answer: Bernie should receive $\frac{89\,500}{193\,500} \times 234\,000$ = \$108,233 (to the nearest dollar), so Bronwyn should receive 234 000 – 108 233 = \$125,767.)

Examples: Worksheet (Material Master 8–29).

Understanding Number Properties: John and June buy a house together. John contributes \$$g$, and June contributes \$$h$. When they sell the house, they receive \$$t$ altogether. How would you work out how much John should receive as his share? ***(Answer: \$ $\frac{g}{g+h} \times t$.)***

Inverse Ratios

Inverse ratio problems are challenging and require careful teaching.

I am learning how to solve inverse ratio problems.

Equipment: Worksheet (Material Master 8–30).

Using Number Properties

Problem: "Jane, Julie, and Atanga plan to paint a house in 24 days. Jane gets sick, so Julie and Atanga have to paint the house without her. How long will it take the pair of them to do the painting?"

Discuss how long 1 person would take to do the job alone.

(Answer: If 3 people take 24 days, 1 person takes 3 times as long = $3 \times 24 = 72$ days.)

Discuss how long 2 people would take.

(Answer: $72 \div 2 = 36$.)

Examples: Worksheet (Material Master 8–30).

Understanding Number Properties: It takes z painters y days to paint a house. How long will it take v painters? ***(Answer:** $\frac{zy}{v}$ **days.)***

Hard Problem: "3 friends paint 231 bicycles in 7 days. How long will it take 8 friends to paint 616 bicycles?"

(Answer: $3 \times 7 = 21$. So 21 "friend-days" of work produces 231 bicycles. So 1 "friend-day" of work produces $231 \div 21 = 11$ painted bicycles. So 8 friends will paint 88 bicycles per day. So 616 bikes will take $616 \div 88 = 7$ days.)

Understanding Number Properties (hard): It takes z painters y days to paint n houses. How long will it take v painters to paint r houses? ***(Answer:** $\frac{zyr}{vn}$ **days.)***

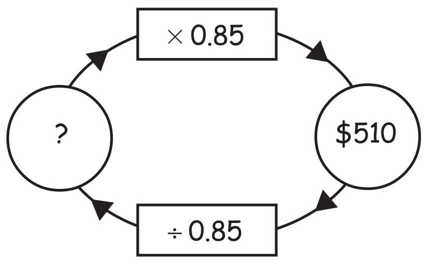
Reverse Percentage Problems

Students with poor understanding of the meaning of percentages find inverse percentage problems impossible. For example, finding the pre-discount price when, after a 15% discount, a TV costs $350 is a challenging problem. Problems like this are easier to solve when percentages are converted to decimals before calculation.

I am learning to undo percentage calculations by reversing flow charts.

Equipment: Calculators. Flow charts (Material Master 8–31).Worksheet (Material Master 8–32).

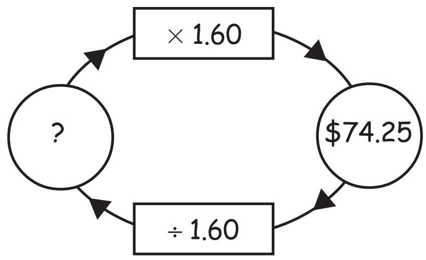
Using Materials



Problem: "Pare receives a 15% discount on a TV, and he pays $510. What was the price *before* the discount?"

Draw an empty flow chart on the board. Discuss why the *unknown* full price is multiplied by 0.85.

Discuss why the full price is found by division, and have the students calculate this price. *(Answer: $510 \div 0.85 = \$600$.)*



Problem: "Melissa buys blouses for her clothes store. She marks the price up by 60%. If she sells a blouse for $74.25, how much did the blouse cost her?"

Discuss why the flow chart shows that the cost price was $74.25 \div 1.60 = \$46.41$ (to the nearest cent).

Using Imaging

Problem: "Janice adds GST of 12.5% to the price of a toaster. It now costs $31.50. What was the price of the toaster before GST was added?"

Imagine the flow chart and reverse it to find the answer. *(Answer: $\$31.50 \div 1.125 = \$28.00$.)*

Examples: Imagine flow charts. Fold back to drawing them if needed. Worksheet (Material Master 8–31).

Using Number Properties

Examples: Worksheet (Material Master 8–32).

Understanding Number Properties: k% of the selling price $\$p$ is profit. What is the cost price? ***(Answer:** $\$\frac{p(100-k)}{100}$**.)***

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AA
AM
AP

Inflation

I am learning about the compounding effect of inflation on prices.

Equipment: Calculators.

Using Number Properties

Problem: "Today the price of an ice cream in the country of Nuldova is $1.50. Prices are rising at the rate of 100% per year. What will be the price of the same kind of ice cream in 20 years' time?"

Discuss why 100% increase means doubling the price each year.

(Answer: 100% increase means the price is 200% of the original, that is, double.)

Discuss why prices are $1.50, $3, $6, $12 ... and how to enter this on a calculator.

(Answer: Enter 1.50 × 2 = = = = counting 20 equal signs, which gives $1,572,864.) Note: Some calculators require the user to go: 2 × 1.5 = = = = ... Carefully check which system your calculators use before attempting this problem.)

On some calculators, you can also use the y^x button to get the answer.

(Answer: Enter 1.50 × 2 y^x 20 = into the calculator.)

Example: A new car currently sells for $20 000. If annual inflation runs at 10% per year, what could you expect an equivalent car to cost in 40 years' time?

(Answer: 20 000 × $(1.1)^{40}$ ≈ $905,185.)

Problem: "In a country where inflation is running wild at 56% per year, how long will it take for an ice cream that currently costs $1 to cost $1,000,000?"

Discuss why entering 1.56 as a constant multiplier in a calculator is sensible. Get students to enter 1.56 × = = = = and count carefully.

(Answer: If, for example, the equals button has been pressed 4 times, the answer to 1.56^5 is in the display not 1.56^4. When 1.56^{31} has been entered, the display shows 970202.8566, and when 1.56^{32} has been entered, the display shows 1513516.456. So the cost reaches $1,000,000 in just over 31 years.)

Examples: For these rates of inflation, find out how long it will take a $1 ice cream to cost $100,000,000: 99% 87% 300% 350% 1 000%

Understanding Number Properties: If inflation is running at 10%, Gerry argues prices will double in 10 years. Explain why Gerry thinks this. Explain why Gerry is wrong. ***(Answer: Gerry says 10 × 10% = 100%, so the price increases by itself, that is to say doubles, in 10 years. But he is wrong. Adding 10% to, say, $1 gives a price of $1.10. For the next year, 10% is added to $1.10 not $1 to give $1.21. So the 10% added each year is being added to an increasing amount every year, so prices will double in less than 10 years.)***

50% on is Not the Same as 50% off!

Apparent contradictions can occur in percentage calculations when the percentages are calculated on the wrong quantity.

I am learning that percentage calculations are related to the base that is used.

Equipment: Calculators. Worksheet (Material Master 8–33).

Using Number Properties

Newtown

Population 1983: 20 000
Population 2003: 25 000

Problem: "Sally notices Newtown's population has increased from 1983 to 2003. She claims the percentage increase is 25%. Samuel claims the increase is 20%. Which one is sensible?"

(Answer: Sally will argue the increase is 5 000 people. Comparing this to the 1983 population, this is a fractional increase of $\frac{5\,000}{20\,000}$ = 0.25 = 25%. Samuel will argue the increase is 5 000 people, but comparing this to the 2003 population, this is a fractional increase of $\frac{5\,000}{25\,000}$ = 0.2 = 20%. Both calculations are correct but it makes sense to calculate increases on the 1983 population. So Sally's answer is better.)

EA
AA
AM
AP

Problem: "Harry buys cans of baked beans from the wholesaler to sell at his dairy. The wholesaler's card says that Harry will make 35% profit. Harry calculates 1.35 × 1.30 and gets 1.755 not 2.00. Explain what he has done wrong."

Baked Beans
Profit 35%
Wholesale: $1.30
Retail $2.00

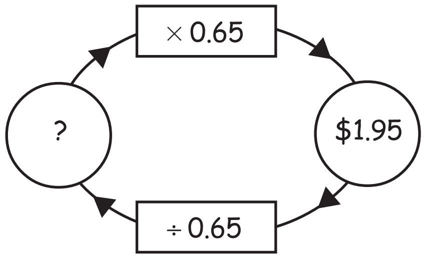
(Answer: This is tricky. The wholesaler reasonably argues that Harry should be interested in profit as 35% of the money he gets in his till, not of the amount he paid for the baked beans. So the 35% profit refers to a percentage of selling price ***not*** *cost price.)*

"Check that the wholesaler is correct."

(Answer: 0.65 × $2.00 = $1.30 is correct.)

Problem: "Harry buys packets of chocolate biscuits at the wholesaler for $1.95 each. He wants to make 35% profit on the selling price. What should Harry charge for a packet of biscuits at his shop?"

Discuss how to build up the flow chart and why Harry should charge 1.95 ÷ 0.65 = $3.00 for a packet of biscuits.



(Answer: 35% of the unknown selling price that needs to be multiplied is gross profit, so 65% of this price must be used to pay the wholesaler. The flow chart shows ? × 0.65 = $1.95. So, reversing the flow chart, the selling price of a packet of biscuits is 1.95 ÷ 0.65 = $3.00.)

Examples: Worksheet (Material Master 8–33).

Understanding Number Properties: Harry decides to make the profit 40% on the selling price of the goods at his store. Make up a problem with this new mark-up and solve it.

Understanding Number Properties (hard): Harry decides to make the profit r% on the selling price of goods that he pays $$p$ for. What price does he charge? ***(Answer:*** $\frac{p}{1-\frac{r}{100}}$***.)***

GST Rules

In New Zealand, the current rate of GST is 12.5% after initially being 10%. For these percentages (and only a few others), there are two simple whole number division rules, one to add on GST, and the other remove GST.

I am learning that there are simple division rules to add on and subtract tax for a few rates of GST.

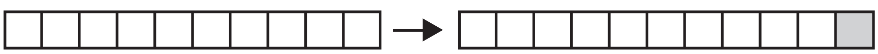
Equipment: Rulers. Squared paper.

Using Materials

Problem: "When GST began in New Zealand, the rate was 10%. When a price includes tax, what fraction of it is tax?"

"What is 10% as a familiar fraction?" *(Answer:* $\frac{1}{10}$*.)*

Draw a rectangle 20 centimetres by 2 centimetres to represent a number and divide it into 10 parts. Add a shaded tenth to represent GST.



"Look at the new diagram as a *new* whole. What fraction is shaded?" *(Answer:* $\frac{1}{11}$*.)*

Discuss why adding 10% to a number is reversed by subtracting $\frac{1}{11}$ from the answer. ($\frac{1}{11}$ of the GST inclusive price is tax.)

Examples: Find the fractions of GST inclusive prices that are tax for these rates of GST. Draw pictures: 12.5%   20%   25%   5%

Using Number Properties.

Problem: "Herbie pays $98.55 for a new pair of trousers. The bill says this amount includes 12.5% GST of $10.95. Is this correct?"

(Answer: Yes 12.5% = $\frac{1}{8}$. So a GST inclusive price has $\frac{1}{9}$ tax. And $\frac{1}{9}$ of $98.55 = $10.95.)

Examples: The following prices include GST for a given GST rate. Find the amount of tax included in the prices: $495, GST 10% $900, GST 12.5% $120, GST 20% $1,000, GST 25%

(Answer: $\frac{1}{11}$ of 495 = $45) *(Answer: $\frac{1}{9}$ of 900 = $100)* *(Answer: $\frac{1}{6}$ of 120 = $20)* *(Answer: $\frac{1}{5}$ of 1 000 = $200)*

Understanding Number Properties: For a GST rate of 50%, explain how to find the tax included in a GST-inclusive price. (***Answer: multiply by $\frac{1}{3}$ or divide by 3.)***

Understanding Number Properties: For a GST rate of $\frac{100}{n}$ %, explain what fraction of the GST-inclusive price is tax. (***Answer:*** $\frac{1}{n+1}$.)

Pigeonholes

The "pigeonhole" principle is a very curious idea that enables mathematicians to solve some otherwise impossible problems.

I am learning to apply the "pigeonhole" principle.

Equipment: Pieces of card. Counters.

Using Materials

Problem: "A postie is sorting letters into pigeonholes at the sorting centre. She has 4 letters to sort into 3 pigeonholes. She is not a very good postie, so she decides to throw the letters into the pigeonholes at random." Get students to use 4 counters and 3 pieces of card to represent letters and pigeonholes respectively. A mathematician says the postie will put 2 or more letters in someone's pigeonhole. Explore this claim with counters.

(Answer: It is correct. Trying to avoid putting 2 in any pigeonhole leads to 1 in each. But the fourth letter now has to go in to someone's pigeonhole.)

Examples: Convince yourself that these claims are true using counters:
For 5 letters and 3 pigeonholes, someone gets 2 letters or more.
For 6 letters and 3 pigeonholes, someone gets 2 letters or more.
For 7 letters and 3 pigeonholes, someone gets 3 letters or more.

Understanding Number Properties: Find the value of k in each case:
For 23 letters and 3 pigeonholes, someone gets k letters or more. ***(Answer: k = 8)***
For 17 letters and 4 pigeonholes, someone gets k letters or more. ***(Answer: k = 5)***

Using Number Properties

Problem: "Convince yourself that these claims are true without using materials:
For 12 letters and 5 pigeonholes, someone gets 3 letters or more.
For 24 letters and 5 pigeonholes, someone gets 5 letters or more.
For 101 letters and 5 pigeonholes, someone gets 21 letters or more.

Discuss why these all work."

(Answer: To minimise the number someone must receive, distribute the same number of letters into each pigeonhole. This is done by division. If there is no remainder, then the quotient is the minimum number. However, if there is a remainder, someone must receive another letter, so the answer is the quotient plus 1.)

Examples: Find k in each case:
For 80 letters and 10 pigeonholes, someone gets k letters or more.
For 101 letters and 10 pigeonholes, someone gets k letters or more.
For 45 letters and 8 pigeonholes, someone gets k letters or more.
For 1 001 letters and 4 pigeonholes, someone gets k letters or more.

EA
AA
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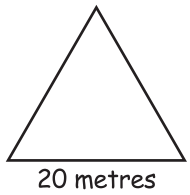

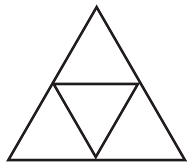

Problem: "A very mathematically inclined farmer encloses an area of grass for his cows by making an equilateral triangle with electric fencing wire. The sides of the triangle are 20 metres long. The farmer places 5 cows within the fence. After watching for some time, he notices something unusual. No matter how the cows arrange themselves at least 2 of them are within 10 metres of each other at all times. Why is this always correct?"

Hint: Add lines to create 4 equilateral triangles and experiment with placing cows (5 counters) so no triangle has 2 cows in it.

(Answer: In trying to prevent 2 going in any triangle, you can place 1 in each triangle. But now the fifth cow must go in one of the triangles that already has a cow in it.)

Examples: If the equilateral field has a side of 30 metres draw a diagram and explain why having 10 cows in the field means 2 of them are within 10 metres of each other at all times.

If the equilateral field has a side of 40 metres, draw a diagram and explain why having 17 cows in the field means 2 of them are within 10 metres of each other at all times.

Understanding Number Properties: The farmer builds an equilateral triangle with sides 10n metres long. How many cows must he put in the enclosed area to ensure there is a pair within 10 metres of each other at all times. ***(Answer: $n^2 + 1$ cows.)***

Systematic Prime Factorisation

A systematic procedure to factorise a number is to test whether or not any prime number up to and including the square root of the number is a factor. If there is no such prime factor then the number is a prime. If there is such a factor the procedure can be reapplied to the new number derived from the number divided by the prime factor. This procedure continues until a number which is prime emerges.

I am learning a method to prime factorise any number.

Equipment: Calculators.

Using Number Properties

Jules tries to factorise 16 709. He uses divisibility rules and sees 2, 3, 5 are not factors of 16 709. Why does he not test whether 4 and 6 are factors?

(Answer: If 2 is not a factor then 4 cannot be a factor. If 2 is not a factor then 6 cannot be a factor.)

Why does Jules test for 7 next?

(Answer: 7 is the next prime number after 5.)

Jules works out 16 709 ÷ 7 = 2387 on a calculator. He searches for a prime factor of 2387. Does he need to check 2, 3, 5 or 7 again?

(Answer: He does not need to check 2, 3 or 5 as they are not factors of the original number. But 7 is a factor of 16 709, and so it might also be a factor of 2387.)

Explain how Jules knows 16 709 factorises in primes to 7 × 7 × 11 × 17.

(Answer: 2387 ÷ 7 = 341 so 7 is again a factor. Checking 7 again 341 ÷ 7 is not a whole number. The next prime is 11, and 341 ÷ 11 = 31. As 31 is prime the procedure stops. So 16 709 = 7 × 7 × 11 × 31.)

Examples: Find the prime factorisations of these numbers: 646 19 530 527
353 5439 6273 2136 127 400 559 000

Example: Kevin buys packets of biscuits for a school camp. He counts them and finds he has 899. Realistically there are two possible numbers of biscuits in a packet. What are the numbers?

Understanding Number Properties: If g is a factor of a explain why a ÷ g is also a factor of a. Use this to explain this true statement: If a has no factors that are less than or equal to $\sqrt{a}$, then a is a prime number.

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