Answers and Teachers' Notes

MINISTRY OF EDUCATION
Te Tāhuhu o te Mātauranga

Contents

Introduction

This book is one of three in the Figure It Out series that have proportional reasoning as their focus: *Proportional Reasoning*, Level 3+ and *Proportional Reasoning*, Level 3–4+ (Books One and Two). In these books, students explore the meaning of fractions and ratios and learn how to use them to make comparisons in a wide variety of contexts.

The books have been developed to support teachers whose students are moving into the early proportional and advanced proportional stages of the number framework (stages 7 and 8). Like the other "plus" books in the Figure It Out series, these should be suitable for students needing extension. The level 3–4+ books are intended for use with students in year 6 but could be used at other levels at the discretion of the teacher.

The books aim to set activities in real-life and imaginary contexts that should appeal to students. The real-life contexts reflect many aspects of life in New Zealand, and the young people portrayed in illustrations and photos reflect our ethnic and cultural diversity.

The activities may be used as the focus for teacher-led lessons, for students working in groups, or for independent activities. But bear in mind that the Figure It Out series is a resource, not a set of textbooks. This means that if you are setting an activity to be done independently, you should check that you have done whatever prior teaching is needed.

Teachers sometimes say that their students have difficulty understanding the words on the page. We are very mindful of this and try to keep written instructions as brief and as clear as possible, but to create a context and pose questions, some words must be used. It is important that mathematical language and terminology be deliberately taught.

The Answers section of the *Answers and Teachers' Notes* that accompany each student book includes full answers and explanatory notes. Students can use them for self-marking, or you can use them for teacher-directed marking. The teachers' notes for each activity include achievement objectives, a commentary on the mathematics involved, and suggestions on teaching approaches. **Although the notes are directed at teachers, able students can use them as a self-help resource.** The *Answers and Teachers' Notes* are also available on Te Kete Ipurangi (TKI) at www.tki.org.nz/r/maths/curriculum/figure/

Where applicable, each page starts with a list of the equipment needed. Encourage the students to be responsible for collecting this equipment and returning it at the end of the session.

Encourage your students to write down how they did their investigations or found solutions, drawing diagrams where appropriate. Discussion of strategies and answers is encouraged in many activities, and you may wish to ask your students to do this even where the instruction is to write down the answer.

> The ability to communicate findings and explanations, and the ability to work satisfactorily in team projects, have also been highlighted as important outcomes for education. Mathematics education provides many opportunities for students to develop communication skills and to participate in collaborative problem-solving situations.
>
> *Mathematics in the New Zealand Curriculum,* page 7

Students will have various ways of solving problems or presenting the process they have used and the solution. You should acknowledge successful ways of solving questions or problems, and where more effective or efficient processes can be used, encourage the students to consider their merits.

Figure It Out

Proportional Reasoning
Level 3–4$^+$, Book Two
Answers

Page 1: Running Hot and Cold

Activity

1. a. The right-hand diagram shows what happens. The hot tap fills the spa in less time than the cold tap, which means that the hot water flow rate is greater than the cold water flow rate. So if both are on full, more of the water in the spa will come from the hot tap.

   b. $\frac{3}{5}$ from the hot tap and $\frac{2}{5}$ from the cold tap.

2. a. The cold tap fills $\frac{3}{45} = \frac{1}{15}$ of the spa. The hot tap fills $\frac{3}{30} = \frac{1}{10}$ of the spa.

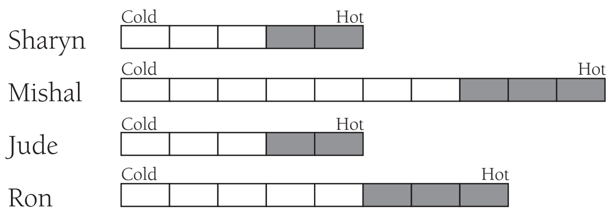
   b. $\frac{1}{6}$. ($\frac{1}{15} + \frac{1}{10} = \frac{2}{30} + \frac{3}{30} = \frac{5}{30} = \frac{1}{6}$)

   c. 18 min. (6 x 3 min)

3. a.

Friend	Cold only (min)	Hot only (min)	Both taps
Sharyn	20	30	$\frac{3}{5}$ x 20 = 12 min
Mishal	15	35	$\frac{7}{10}$ x 15 = 10.5 min
Jude	10	15	$\frac{3}{5}$ x 10 = 6 min
Ron	12	20	$\frac{5}{8}$ x 12 = 7.5 min

   b. Explanations will vary, but they will all involve ratio. You could use the same method as for question **2** above. This strip diagram shows the ratios:



The method used to complete the table above goes like this (using Sharyn's bath as an example): Using ratio to compare the *time taken*, we get 20:30 = 2:3. Reverse this ratio to compare *flow*, and we have 3:2. (The cold takes less time because it flows faster.) This means that $\frac{3}{5}$ of the water in the full bath will have come from the cold tap (and $\frac{2}{5}$ from the hot tap). We know that if the cold tap is running by itself, it will fill the bath in 20 min, so if it is just running for long enough to fill $\frac{3}{5}$ of the bath, it must take $\frac{3}{5}$ x 20 = 12 min.

Pages 2–3: Deb the Driver

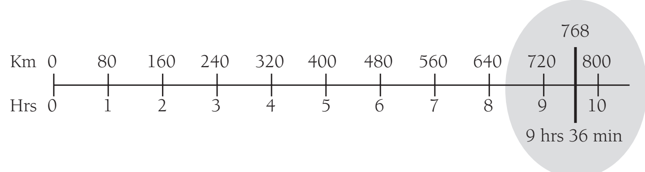
Activity One

1. 359.8 km
2. About $\frac{5}{8}$ of 48 L, which is 30 L
3. Deb's car travelled about 360 km on about 30 L, which is 12 km/L.
4. The car has done 96.1 km on about $\frac{1}{8}$ of 48 L. That is about 96 km, using approximately 6 L. This is 16 km/L, so yes, Deb's fuel consumption has improved.

Activity Two

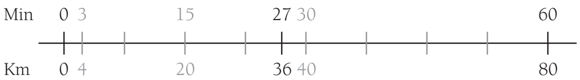
1. a. The tank holds 48 L and the car now travels 16 km for each litre of fuel, so Deb should be able to drive 48 x 16 = 768 km on a tankful.

   b. In 1 hr, Deb travels 80 km. So she can drive for 768 ÷ 80 = 9$\frac{3}{5}$ hrs (9 hrs and 36 min). This can be shown on a number line:



2. 90 km/h means that the car travels 90 km in 60 min, which is 1$\frac{1}{2}$ km each min. To cover 36 km should take 24 min. (24 x 1$\frac{1}{2}$ = 36)

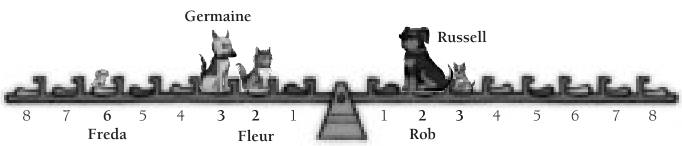
3. If Deb covers 36 km in 27 min, she covers 4 km every 3 min (as on the number line shown), or 8 km every 6 min, or 80 km every 60 min. This is a speed of 80 km/h.

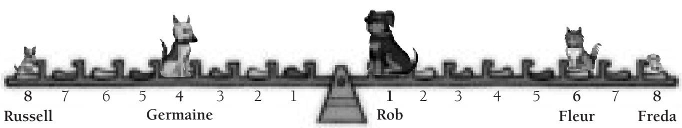


Pages 4–5: Dog Torque

Activity

1. Seat 2. (16 x 3 = 8 x 5 + 4 x 2)
2. Seat 3. (32 x 5 + 4 x 8 = 64 x 3)
3. The closer the animal is to the end of the see-saw, the greater the effect of its weight (the greater the torque it exerts). A dog weighing 8 kg sitting in seat 4 exerts the same torque as a dog weighing 16 kg sitting in seat 2. (8 x 4 = 16 x 2)
4. Answers will vary, but they must match the torque being exerted by Germaine (32 x 3 = 96 units). Possible answers include:
   - Fleur at seat 2 and Russell at seat 8 (16 x 2 + 8 x 8 = 96)
   - Rob at seat 1 and Fleur at seat 2 (64 x 1 + 16 x 2 = 96)
   - Fleur at seat 6 (16 x 6 = 96)
   - Rob at seat 1 and Freda at seat 8 (64 x 1 + 4 x 8 = 96)
   - Freda at seat 2, Fleur at seat 3, and Russell at seat 5 (4 x 2 + 16 x 3 + 8 x 5 = 96)
5. Possible answers include:





6. a. Rob can sit in seat 6, but not in seats 7 or 8. (The maximum torque that can be exerted by the other dogs is 8 x 32 + 7 x 16 + 6 x 8 + 5 x 4 = 436. 436 ÷ 64 [64 is Rob's mass] = 6.8, showing that Rob can't sit further back than seat 6.)

   b. A possible answer is Germaine at seat 8, Fleur at seat 7, Freda at seat 2, and Russell at seat 1.
7. Possible answers include:
   - End 1: Larry in seat 1; End 2: Freda in seat 6.
   - End 1: Larry in seat 4 and Russell in seat 6; End 2: Freda in seat 4, Germaine in seat 3, and Fleur in seat 2.
   - End 1: Germaine in seat 8, Larry in seat 7, Freda in seat 6, and Fleur in seat 5; End 2: Russell in seat 2 and Rob in seat 8.
8. Answers will vary. Many problems are possible.

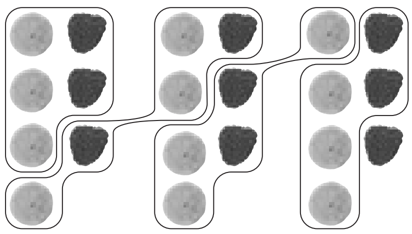
Pages 6–7: Flavoursome

Activity One

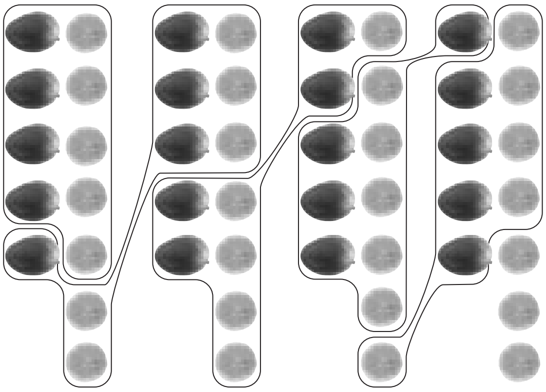
1. The spare measure tells Whina that the 5:3 recipe has proportionally more apple juice in it than the 3:2 recipe. This means that the 5:3 recipe will have a stronger apple taste and the 3:2 recipe will have a stronger blueberry taste.
2. Whina wants to match either the amounts of apple juice or the amounts of blueberry juice. By doubling the 5:3 recipe and trebling the 3:2 recipe, she makes the blueberry match (6 measures). If she takes 3 copies of the 5:3 recipe (as suggested) and 5 copies of the 3:2 recipe, she will match the amounts of apple juice. (She will then discover that 3 copies of the first recipe [15:9] will "fit" into the 5 copies of the second recipe [15:10], leaving one "spare" measure of blueberry juice.)

Activity Two

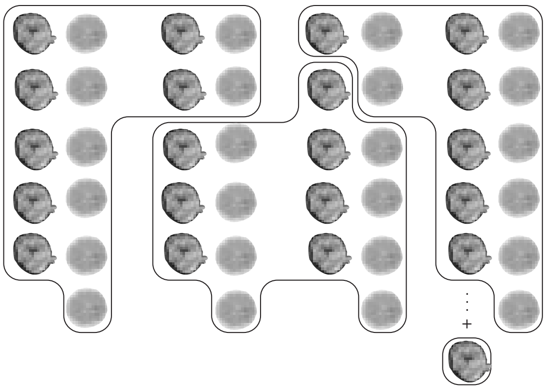
1. If recipe A is cloned 3 times, recipe B will "fit into" it 4 times, with 1 measure of raspberry left over. This means that recipe A has a stronger raspberry flavour than recipe B.



2. Recipe D can be cloned 4 times to give a 20:28 mixture. Cloning recipe C 5 times gives 20:25; when the second mixture is fitted into the first, 3 measures of orange are left over. This shows that recipe C has a stronger mango flavour than recipe D.



3. Recipe E could be cloned 4 times to give 20:24 and recipe F cloned twice to give 14:16. A third clone of recipe F could be obtained if there were just 1 more measure of passionfruit, meaning that recipe F must have a stronger passionfruit flavour than recipe E.



Page 8: Just Right

Activity

1. Terry's recipe is equivalent to 1 tbs of powder for just over 60 mL of milk, and Tracey's recipe is 1 tbs for 50 mL of milk. So Tiana's drink should be made using a ratio of 1 tbs of powder for about 55 mL of milk. For a similar sized drink, the recipe might be 5 tbs of powder for 275 mL of milk.

2. a. Terry's recipe is 2:11 or 1 part orange cordial to $5\frac{1}{2}$ parts of water. Tracey's recipe is 1:5 or 1 part orange cordial to 5 parts of water. So Tiana's drink should be between 5 and $5\frac{1}{2}$ parts of water per part of orange cordial. A suitable recipe would be $5\frac{1}{4}$ parts of water per part of orange cordial (or 21 parts of water per 4 parts of orange cordial).

   b. Terry's recipe is 6 tsp of powder for $\frac{3}{4}$ glass of milk, which is the same as 8 tsp of powder for one full glass of milk. Tracey's recipe is 5 tsp of powder for $\frac{1}{2}$ glass of milk, which is the same as 10 tsp for a full glass of milk. A suitable recipe would be 9 tsp of powder for a full glass of milk.

Page 9: Fully Grown

Activity

1. 180 cm (1.8 m)
   If 144 cm is 80% of David's adult height, 10% must be 144 ÷ 8 = 18 cm, so 100% must be 18 x 10 = 180 cm.

2. 90 cm (0.9 m)
   If 180 cm is David's likely adult height, his likely height at age 2 was 50% of this.
   50% of 180 = 90 cm.

3. 153 cm at 12 yrs (85% of 180)
   165.6 cm at 14 yrs (92% of 180)
   178.2 cm at 16 yrs (99% of 180)

4. Angie is 9 cm taller than David.
   Fully grown, David is likely to be 180 cm, so Angie's adult height is likely to be 178 cm.
   At age 10, her height should be 86% of her adult height, that is, 86% of 178. 0.86 x 178 = 153 cm, which is 153 – 144 = 9 cm taller than David is right now.

Pages 10-11: Ratio Rip

Game

A game that involves matching ratios and fractions

Pages 12-13: Laser Blazer

Activity

1. a. \$10. (They paid for 4, and 2 got in free. 4 x 15 = \$60. 60 ÷ 6 = \$10.)

b. $33\frac{1}{3}$%. (Without the discount, they would have paid 6 x 15 = \$90, so they saved \$30. 30 ÷ 90 = $\frac{1}{3}$ or $33\frac{1}{3}$%.)

2. Luke $\frac{30}{50}$ = 60%
Tangihaere $\frac{30}{40}$ = 75%
Matt $\frac{42}{56}$ = 75%
Alex $\frac{44}{66}$ = $66\frac{2}{3}$%
Ese $\frac{36}{60}$ = 60%

3. a. Luke $\frac{12}{20} = \frac{3}{5}$
Tangihaere $\frac{21}{24} = \frac{7}{8}$
Matt $\frac{30}{40} = \frac{3}{4}$
Alex $\frac{11}{22} = \frac{1}{2}$
Ese $\frac{17}{17}$ (all)

b. Ese. To gain marksman rating, he will have to score hits with all of his remaining 17 shots.

Pages 14–15: The Percentage Game

Game

Answers (clockwise from Start):
28 is **40%** of 70
16 is **25%** of 64
40% of **26** is 10.4
19 is **20%** of 95
40% of **28** is 11.2
26 is **50%** of 52
18 is 40% of 45
30% of **95** is 28.5
48 is **80%** of 60
9 is **45%** of 20
16 is 50% of 32
24 is **30%** of 80
36 is **60%** of 60
20% of **90** is 18
72 is 75% of 96
18 is **10%** of 180

Page 16: A Sizeable Problem

Activity

1. a. The 150% button

b. The 50% button

c. Askaz thought that increasing her height by 50% then decreasing the result by 50% would get her back to her normal height. But 50% of her increased height returns her to just 75% of her normal height. She should have pressed the 67% ($\frac{2}{3}$) button.
(150% x 67% = 100% or $\frac{3}{2} \times \frac{2}{3} = 1$)

2. a. The 125% button and the 80% button.
(125% x 80% = 100% or $\frac{5}{4} \times \frac{4}{5} = 1$)

b. The 133% button and the 75% button.
(133% x 75% = 100% or $\frac{4}{3} \times \frac{3}{4} = 1$)

c. The 75% button and the 133% button.
(75% x 133% = 100% or $\frac{3}{4} \times \frac{4}{3} = 1$)

d. The 80% button and the 125% button.
(80% x 125% = 100% or $\frac{4}{5} \times \frac{5}{4} = 1$)

e. The 200% button and the 50% button.
(200% x 50% = 100% or $2 \times \frac{1}{2} = 1$)

3. There is only one strategy, and that is to multiply by the inverse. But this strategy can be explained in different ways and using different examples.
Here is one explanation:
If Yonaz increases his height by $\frac{1}{3}$, his height is multiplied by $1 + \frac{1}{3} = \frac{4}{3}$. To get back to his normal height, he needs to multiply his (increased) height by the inverse of $\frac{4}{3}$, which is $\frac{3}{4}$.

Page 17: Pay Rates

Activity One

1. Kylie gets more pay, but Karen gets paid more per hr. (Kylie earns 27 ÷ 6 = \$4.50/hr; Karen earns 20 ÷ 4 = \$5.00/hr.)

2. a. Kylie is paid \$4.50/hr and Karen \$5.00/hr, so Mrs White must be paying
4.50 + 5.00 = \$9.50/hr.
200 ÷ \$9.50 = 21.05 hrs, so Mrs White must be paying the girls for 21 hrs each, at a total cost of 21 x 9.50 = \$199.50.

b. Kylie is paid 21 x 4.50 = \$94.50 and Karen is paid 21 x 5.00 = \$105.00.

Activity Two

1. Karen gets \$5.50. (5.00 + 0.50 = \$5.50)
Kylie gets \$5.40. (4.50 + 0.90 = \$5.40)

2. 18 hrs.
(5.50 + 5.40 = $10.90. 200 ÷ 10.90 = 18.35 hrs, or 18 whole hrs.) Karen gets $99.00 (18 x 5.50 = $99.00). Kylie gets $97.20 (18 x 5.40 = $97.20).

3. 8%. (Karen needs to increase her pay from $5.00 to $5.40 to match Kylie's new rate. This is an increase of 40c for every $5.00 or 8c for every 100c, which is 8%.)

Pages 18–19: The Equivalence Game

Game

A game using equivalent fractions

Pages 20–21: The Right Gear

Activity

1. a. 2 times

   b. The top wheel will now have to turn twice to make the bottom wheel turn once.

   c. The big wheel should be attached to the main shaft.

   d. Fastest: 48-tooth wheel on the main shaft and the 12-tooth wheel on the dynamo.
   Slowest: 12-tooth wheel on the main shaft and the 48-tooth wheel on the dynamo.

2. 80 times. ($\frac{48}{12}$ = 4. 20 x 4 = 80)

3.

Wind (turns of blades)	Large wheel (teeth)	Small wheel (teeth)	Dynamo (turns)
20	48	16	60
30	36	12	**90**
26	**36**	18	52
48	36	24	72
21	48	**16**	63

4. Three combinations are possible:

Blades turn 21 times per min		
Teeth	**36**	**48**
12	63	84
16	47.25	63
18	42	56
24	31.5	42

Challenge

The best performance comes from using the 36-tooth wheel together with the 16-, 18-, and 24-tooth wheels:

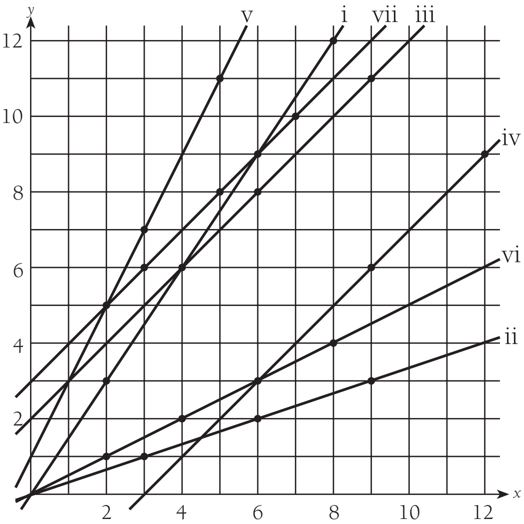
Small wheel (teeth)	Blade speed (turns per min)																				
	25	**26**	**27**	**28**	**29**	**30**	**31**	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**	**40**	**41**	**42**	**43**	**44**	**45**
	Dynamo speed (turns per min) with 36-tooth wheel attached to the main shaft																				
12																					
16	56	59	61	63																	
18				56	58	60	62	64	66	68	70										
24											53	54	56	57	59	60	62	63	65	66	68

Note that Fred can't use the 48-tooth wheel because, even at its lowest gearing (using the 24-tooth wheel on the dynamo), it can only provide the right performance in winds that turn the blades at up to 32 times per min.

Pages 22–23: Family Likeness

Activity

1. **a.**



b. Answers will vary.

c. (0,0) belongs to sets i, ii, and vi.

2. Note that these answers can be expressed in different ways:
   - **i.** Multiply the first number by $1\frac{1}{2}$ to get the second number.
   - **ii.** Divide the first number by 3 to get the second number.
   - **iii.** Add 2 to the first number to get the second number.
   - **iv.** Take 3 off the first number to get the second number.
   - **v.** To get the second number, double the first number and add 1.
   - **vi.** Halve the first number to get the second number.
   - **vii.** Add 3 to the first number to get the second number.

3. (18,6) belongs with set ii; (20,10) belongs with set vi; (14,17) belongs with set vii; (14,21) belongs with set i; (11,23) belongs with set v; (17,14) belongs with set iv; (13,15) belongs with set iii.

4. Answers will vary. Here is one example:
   {(0,0), (3,15), (4,20)}

Page 24: da Vinci's Ratio

Investigation One

1. Practical activity. Results will vary.

2. **a.** A "1" in this column would mean that the person's arm span and height measurements are exactly the same. If the number is less than 1, the person's arm span is less than their height. If the number is greater than 1, their arm span is greater than their height.

   b. The ratios should be similar – all quite close to 1 – showing that da Vinci's conclusion is close to the truth.

   c. da Vinci might explain that, as with any real-life data, there will be variation.

   d. Graphs will vary, but the clustering of points about an imaginary straight line should show clearly that a connection does exist between arm span and height.

Investigation Two

1. Practical activity. Results will vary. There is a relationship between head size and height. (We would certainly notice if someone's head seemed too big or too small for their body.) But the relationship is not exact, and it changes from birth through to when growth stops in the late teens. For a mature adult, their height is typically about 3 times the circumference of their head. For a child of early school age, their height is typically about 2.2 times the circumference of their head.

2. Answers will vary. Using an adult height:circumference ratio of 3:1, the head circumferences should be approximately:

   a. 52 cm

   b. 61 cm

   c. 55 cm.

Figure It Out

Proportional Reasoning

Teachers' Notes

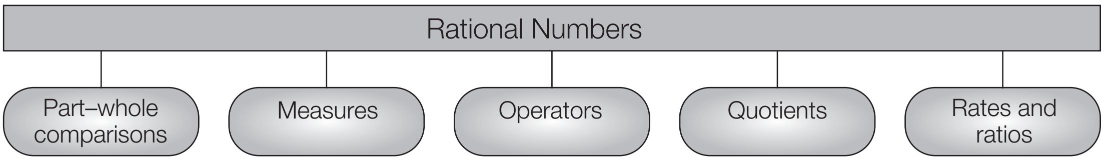
Overview of Level 3–4+: Book Two

Introduction

It is said that the rhinoceros beetle is the strongest animal on Earth; by some accounts, it can support up to 850 times its own weight on its back – the equivalent of a man supporting 75 cars. This does not mean that a rhinoceros beetle can lift heavier objects than any other animal; rather, that it is *proportionally* stronger than any other animal: the fairest measure of its strength is found by comparing what it can lift with its body weight.

Before they can make multiplicative comparisons of this kind, students need to extend their knowledge of numbers to include all rational numbers (those that can be written as fractions in the form $\frac{a}{b}$). With their two components (numerator and denominator), these numbers are able to express the relationship between two measures.

A difficulty for both teacher and student is that rational numbers can be used and interpreted in subtly different ways depending on the context. Kieran[1] suggested this helpful classification:



1. *Part–whole* comparisons involve finding the multiplicative relationship between part of a continuous space or of a set and the whole. For example, what fraction of a square has been shaded?

2. In a *measurement* context, a rational number is the answer to questions of the kind, "How many times does this fraction (or ratio) fit into that fraction (or ratio)?"

3. As *operators*, rational numbers perform operations on other numbers, for example, $\frac{1}{3} \times 12 = \square$.

4. As *quotients*, rational numbers provide the answers to sharing problems. It is important for students to recognise that $7 \div 4$ is an *operation* while $\frac{7}{4}$ (the quotient) is a *number* that is the result of that operation.

5. *Rates* involve a multiplicative relationship between two variables, each with a different unit of measurement (for example, kilometres and hours). Ratios are a special case of rates in that the units of measurement are the same for each variable (for example, 1 shovel of cement to every 5 shovels of builders' mix).

It is important that students are exposed to rational numbers in all their guises and that they learn to attribute different meanings to them, depending on the use and the context. It is also important that students learn a range of different ways of modelling situations that require proportional reasoning. This book will help in both areas. It should also help the teacher recognise that many everyday contexts can provide relevant and often intriguing rate and ratio challenges for their students.

[1]Kieran, T. (1976), "On the Mathematical, Cognitive, and Instructional Foundations of Rational Numbers", in Lesh, R. (ed.), *Number and Measurement: Papers from a Research Workshop*, Columbus, Ohio: ERIC/SMEAC.

Page 1: Running Hot and Cold

Achievement Objectives

- solve practical problems which require finding fractions of whole number and decimal amounts (Number, level 3)
- find a given fraction or percentage of a quantity (Number, level 4)

AC
EA
AA
AM
AP

Number Framework Links

Use this activity to:

- help students consolidate and apply their knowledge of equivalent fractions (stage 7)
- develop confidence in students who are beginning to use advanced proportional strategies (stage 8).

Activity

This activity explores ratio in the context of the time taken to fill a spa. To answer the questions, students need to come to understand the inverse relationship that exists between flow rate and time.

Ask the students to discuss question **1** in pairs and then decide how they might explain their answers in a way that is likely to convince someone else. Get them to report back to the larger group. Make sure that they have an answer for the question *"Why is the cold:hot time ratio (3:2) the reverse of the cold:hot volume ratio (2:3)?"* If they are not familiar with the concept of inverse relationships, discuss them at this point and get the students to come up with other examples.

The parts of question **2** have been designed to guide the students through a strategy that will lead to a solution for part **c**. The same strategy can then be used to solve question **3**.

Students may find that drawing a double number line for each tap will help them see the relationship between time and water level:

Cold tap only

Min	0	3	6	9	12	15	30	45
Fraction filled	0	$\frac{1}{15}$				$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{3}$ or 1

Hot tap only

Min	0	3	6	9	12	15	30
Fraction filled	0	$\frac{1}{10}$				$\frac{1}{2}$	$\frac{2}{2}$ or 1

Question **2b** can be reworded like this: *"If the cold tap fills $\frac{1}{15}$ of the spa in 3 min and the hot tap fills $\frac{1}{10}$ of the spa in 3 min, what fraction of the spa fills when both taps run simultaneously for 3 min?"* Record and clarify responses until everyone is clear that the equation $\frac{1}{15} + \frac{1}{10} = \square$ correctly interprets their strategy. Using number properties: $\frac{1}{15} + \frac{1}{10} = \frac{2}{30} + \frac{3}{30} = \frac{5}{30} = \frac{1}{6}$

For question **2c**, ask: *"We now know that with both taps running, it takes 3 min to fill $\frac{1}{6}$ of the spa. How can we use this information to work out how long it will take to fill the whole spa?"* This is a simple case of direct proportion: the longer the taps run, the fuller the spa becomes. Students could represent the time:fullness ratio using a third double number line or a ratio table:

Min	3	6	9	12	15	?
Fraction filled	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	$\frac{6}{6}$

Go back over the three steps used in question **2**. Ask your students why the question told them to use 3 min as a time interval. It is important that they see that this number was chosen because it was a sensible common factor of 45 and 30. (5 min is also a common factor and could be used, but the fraction work involved is a bit more tricky.) You could also challenge the students to write and solve an equation that shows how much of the spa would fill in 1 min with both taps running: $\frac{1}{45} + \frac{1}{30} = \frac{2}{90} + \frac{3}{90} = \square$ ($= \frac{5}{90} = \frac{1}{18}$).

When doing question **3**, students can use the same steps that were modelled in question **2**. They will need to decide for themselves what sensible common factor they should use for each bath. Suitable factors are: 10 (Sharyn), 5 (Mishal), 5 (Jude), and 4 (Ron). Because the Answers use a different strategy, the calculations are set out here for your convenience:

Friend	**Time unit**	**Fraction filled per time unit**	**Thinking**	**Fill time**
Sharyn	10 min	$\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$	$\frac{5}{6}$ in 10 min is the same as $\frac{1}{6}$ in 2 min	6 x 2 = 12 min
Mishal	5 min	$\frac{1}{3} + \frac{1}{7} = \frac{7}{21} + \frac{3}{21} = \frac{10}{21}$	$\frac{10}{21}$ in 5 min is the same as $\frac{2}{21}$ in 1 min	$10\frac{1}{2}$ min (because $10\frac{1}{2}$ x 2 = 21)
Jude	5 min	$\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$	$\frac{5}{6}$ in 5 min is the same as $\frac{1}{6}$ in 1 min	6 x 1 = 6 min
Ron	4 min	$\frac{1}{3} + \frac{1}{5} = \frac{5}{15} + \frac{3}{15} = \frac{8}{15}$	$\frac{8}{15}$ in 4 min is the same as $\frac{2}{15}$ in 1 min	$7\frac{1}{2}$ min (because $7\frac{1}{2}$ x 2 = 15)

Pages 2–3: Deb the Driver

Achievement Objectives

- solve practical problems which require finding fractions of whole number and decimal amounts (Number, level 3)
- find a given fraction or percentage of a quantity (Number, level 4)

AC
EA
AA
AM
AP

Number Framework Links

Use this activity to:

- encourage the transition from advanced multiplicative strategies (stage 7) to advanced proportional strategies (stage 8)
- develop confidence in students who are beginning to use advanced proportional strategies (stage 8).

Activity One

In this activity, students calculate and compare rates in a motoring context. The numbers have been chosen so that the calculations themselves are fairly straightforward; the challenge is to work out how the different bits of information relate to each other mathematically.

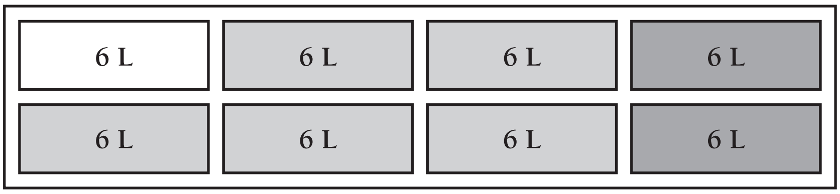
Give your students the opportunity to read the introduction and try to make sense of it for themselves. Possible areas of difficulty are interpreting the scale on the fuel gauge and recognising the place value of the numerals shown on the odometer: it is easy to miss the decimal point.

When they do questions **1** and **4**, encourage the students to estimate the difference in distances before using a calculator. Discuss strategies that might be used. Here is one: "The reading at the start is about 75 km less than 66 000. The reading at the finish is about 285 km greater than 66 000. 75 + 285 = 60 + 300 = 360 km. So Deb drove about 360 km."

Encourage the students to think of different ways of making a good estimate (one that can be done in the head but that gives a clear feeling for what the answer looks like). You could restate the students' methods and help them record them in ways that can be followed. Make a clear distinction between estimates, which point to the answer, and the answer itself.

In question **2**, the students need to work out the difference between the two readings of the fuel gauge. If necessary, they can do this using materials – by folding a piece of paper in half, doing this again, and then folding it a third time. They then unfold the paper to see the eighths and shade the difference between $\frac{7}{8}$ and $\frac{1}{4}$ to reveal the answer of $\frac{5}{8}$.

Students could draw this model of a fuel tank, distribute the 48 L of petrol in 8 equal amounts, and then use the model to calculate the fuel used:



The shading shows that 5 x 6 = 30 L of fuel were used, so $\frac{5}{8}$ of 48 is 30 L.

The students should also look at how the problem could be solved using number properties. If they are able to understand that $\frac{5}{8}$ is 5 x $\frac{1}{8}$, they can find $\frac{1}{8}$ of 48 and then multiply that amount by 5. Alternatively, they could multiply the amount of fuel by 5 (5 x 48 = 5 x 50 – 5 x 2 = 240) and then divide the result by 8. Encourage your students to try both approaches; if they understand both, they will have greater flexibility when it comes to solving other problems involving fractions.

Question **3** introduces the direct relationship that exists between kilometres travelled and litres of fuel used. Check that all your students understand the meaning of "per" and know that it corresponds to the division operation. Note that the question uses km/L as the unit because most students will find this easiest, both mathematically and conceptually. The official unit for measuring fuel consumption is litres per 100 km. Some students may be aware of this. If so, you could challenge them to work out Deb's fuel consumption using this unit: 30 L per 360 km = 30 L per (100 x 3.6) km = (30 ÷ 3.6) L per 100 km = 8.3 L per 100 km.

Students should attempt question **4** independently, applying the ideas discussed previously.

Activity Two

Make sure that the students have the correct answer to the previous question before they begin work on question **1a**. Using it, they should form and solve the equation 16 x 48 = $\square$. They shouldn't need a calculator to do this multiplication; the numbers lend themselves to doubling and halving or partitioning strategies.

Question **1b** needs the correct answer from question **1a**. Encourage your students to take the problem step by step, thinking about the distance Deb will cover in 1 hr, 2 hrs, and so on. If they do this, they will discover that she should be able to drive between 9 and 10 hrs. They will need to use their proportional reasoning skills to work out exactly when the fuel will run out. They could use a double number line as in the Answers, or an equation.

A double number line could be used to solve question **2**, but the numbers are not very convenient. A simpler approach is to use ratio. 90 km are covered in 1 hr, so the distance:time (min) ratio is 90:60 = 30:20 = 3:2. This means that the car should travel 3 km every 2 min, so Deb's car should travel 12 x 3 km in 12 x 2 = 24 min.

Question **3** tells us that Deb's journey actually took 27 min, so her car must have been going slower than 90 km/h. Have the students study the double number line shown and match the information in the diagram with the data from the question. Ask them *"What size are the steps on the number line?"* and *"Why does the number line stop at 40 kilometres in 30 minutes?"* (There's no need to take the line any further because 40 km in 30 min is the same rate as 80 km in 60 min.) An alternative approach is to use ratio, as suggested for question **2**. The distance:time ratio is 36:27, which is 4:3. This means that the car travels 4 km in 3 min, which is the same as 4 x 20 = 80 km in 3 x 20 = 60 min. In other words, the car was doing a speed of 80 km/h.

Pages 4-5: Dog Torque

Achievement Objectives

- write and solve problems which involve whole numbers and decimals and which require a choice of one or more of the four arithmetic operations (Number, level 3)
- find and justify a word formula which represents a given practical situation (Algebra, level 4)

AC
EA
AA
AM
AP

Number Framework Links

Use this activity to develop confidence in students who are beginning to use advanced proportional strategies (stage 8).

Activity

In this activity, students learn to balance distance–weight relationships in the context of a fun application of the Principle of Balanced Torques. In the process, they develop meaning for the concept of an equation.

Definitions:

- *Weight* is used in the student book instead of the more correct *mass*. This is because it is the word that students know and use. Strictly speaking, weight is the gravitational force that the Earth exerts on a mass: Fleur has a mass of 16 kg; her weight is around 160 N (Newtons).
- *Torque* can be thought of as a force that, when applied to a lever (a see-saw is a lever), makes it turn. Torque = the force applied to a lever multiplied by its distance from the fulcrum; it is usually measured in newton-metres (Nm).

Students will know from experience that a light person can balance a heavy person on a see-saw as long as the heavy person sits closer to the fulcrum. This prior knowledge can provide a lead-in to a discussion of the examples given at the start of this activity.

One approach to the activity is to put the students in pairs or small problem-solving groups and see how far they get in understanding the situations presented in the examples. If you do this, get them to report back with their findings before they get too far into the actual questions.

An alternative approach is an introductory whole-group discussion, using questions similar to these to focus the thinking:

- *What do you notice about where the two dogs (Fleur and Germaine) are sitting?*
- *Why is Fleur so far back?*
- *If you compare the weights of the two dogs, what do you notice?*
- *What would happen to the see-saw if Germaine moved one seat forward? Could Fleur then balance the see-saw? Where would she have to sit?*

If your students can't get from the general idea to the mathematical distance–weight relationship, you may find it helpful to use materials. The best piece of equipment is a mathematical balance (equaliser balance), which is available from any of the major equipment suppliers. Using this balance, you can suspend weights at different points along the arms and find by experimentation how to balance a change at one end with an appropriate change at the other.

The key understandings are:

- The closer a dog is to the end of the see-saw, the greater the effect of its weight.
- The torque a dog exerts is found by multiplying its distance from the fulcrum (centre) by its weight.
- If more than one dog sits on the same end of the see-saw, the combined effect of their torques is found by adding their individual torques.
- The torques acting on both sides of the see-saw can be tidily expressed as an equation.

The mathematical ideas are easier for a student to understand than this suggests!

In the first example, Fleur weighs 16 kg and is in seat 8. Germaine weighs twice as much (32 kg) but is only half the distance from the fulcrum (seat 4). This information can be represented in the equation 16 x 8 = 32 x 4. Both sides of the equation equal 128, showing that the torques acting on each end of the see-saw are the same, and the see-saw is balanced. Note that if the torques were not the same, we would not have an equation and would have to use an inequality sign (< or >) to express the relationship mathematically. (For example: 16 x 4 > 8 x 6.)

In the second example, Freda (4 x 8) + Fleur (16 x 6) = Rob (64 x 2)

32 + 96 = 128

128 = 128 ✓

The students can now discuss and record the situations shown in questions **1** and **2**. In question **1**, Fleur exerts a torque of 16 x 3 = 48; Russell exerts a torque of 8 x 5 = 40. To balance the see-saw, Freda will have to sit on Russell's end and exert a torque of 8. As she only weighs 4 kg, she will have to sit in seat 2 (4 x 2 = 8). In question **2**, Freda exerts a torque of 4 x 8 = 32; Germaine exerts a torque of 32 x 5 = 160. Their combined torque is 32 + 160 = 192. Rob weighs 64 kg. To exert of torque of 192, he will need to sit in seat 3 (64 x 3 = 192). Expect your students to write this information in the form of a complete equation.

In question **4**, the starting point is the torque that Germaine exerts in seat 3 (32 x 3 = 96). Encourage your students to come up with a variety of possible solutions, representing each as an equation. Questions **5–8** are all a matter of applying the principles learned and discussed to this point. Students could work in pairs. They will find it helpful to use diagrams and should show that their solutions are correct by writing the appropriate equations.

Pages 6-7: Flavoursome

Achievement Objective

- find fractions equivalent to one given (Number, level 4)

AC
EA
AA
AM
AP

Number Framework Links

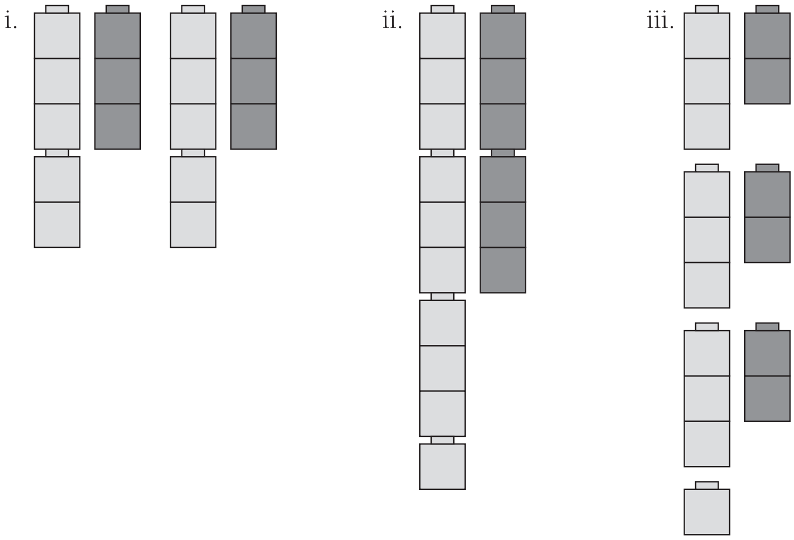
Use this activity to:

- help students consolidate and apply their knowledge of equivalent fractions (stage 7)
- develop confidence in students who are beginning to use advanced proportional strategies (stage 8).

Activity One

In this activity, students learn how to use "cloning": an interesting way of comparing two ratios.

Cloning (alternatively "duplicating" or "replicating") can easily be modelled using multilink cubes as materials, as in the following illustration. A different colour is needed for each part of the ratio:



i. The 5:3 recipe is cloned, and 2 copies made; ii. The 2 copies are combined to make a 10:6 mix; iii. The 10:6 mix is used to make three copies of the 3:2 recipe, with 1 measure of apple juice left over.

Have students clone the ratios and record the results:

Recipe 1	**Recipe 2**
apple:blueberry	apple:blueberry
5:3	3:2
10:6	6:4
	9:6

Ensure that they connect the numbers in the lists with the situation shown in Whina's method.

Students need to be able to make sense of question **1** if they are to be able to do **Activity Two**. The extra apple juice shows that recipe 1 has a stronger apple flavour and conversely that recipe 2 has a stronger blueberry flavour.

Question **2** is designed to highlight the fact that Whina kept cloning both recipes until she found a common amount of either apple or blueberry. Without this, she couldn't compare the ratios. In question **1**, she matched the amounts of blueberry; in question **2**, she matches the amount of apple:

Recipe 1	**Recipe 2**
apple:blueberry	apple:blueberry
5:3	3:2
10:6	6:4
15:9	9:6
	12:8
	15:10

Question **1** shows that recipe 1 has a stronger apple flavour. Question **2** shows that recipe 2 has a stronger blueberry flavour. It is important that your students understand that these are just different ways of stating the same conclusion.

Activity Two

Emphasise the phrase "or your own" to encourage your students to use both cloning (with or without multilink cubes) and another strategy to answer these questions.

A second strategy could be to write the proportion of the named flavour in each recipe as a fraction and then to compare the two fractions with the help of a common denominator. Ask *"What proportions of recipe A and recipe B are raspberry?" "How can we compare these two fractions?"*
In question **1**, recipe A is $\frac{3}{7}$ raspberry and recipe B is $\frac{2}{5}$ raspberry. Using 35 as a suitable common denominator, we can write these two equations: $\frac{3}{7} \times \frac{5}{5} = \frac{15}{35}$ and $\frac{2}{5} \times \frac{7}{7} = \frac{14}{35}$. We can now see that $\frac{3}{7}$ is greater than $\frac{2}{5}$, which shows that recipe A has the stronger raspberry flavour.

Page 8: Just Right

Achievement Objectives

- solve practical problems which require finding fractions of whole number and decimal amounts (Number, level 3)
- find a given fraction or percentage of a quantity (Number, level 4)
- find fractions equivalent to one given (Number, level 4)

AC
EA
AA
AM
AP

Number Framework Links

Use this activity to:

- develop confidence in students who are beginning to use advanced proportional strategies (stage 8)
- help students consolidate and apply their knowledge of equivalent fractions (stage 8).

Activity

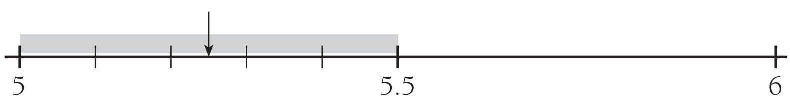
Like the previous activity, this one requires students to compare proportions in the context of drink flavours. But while the previous activity asks them to rank two proportions, this one asks them to find a proportion that is between two others.

If your students have worked through the previous activity, you may wish to give them this one with little introduction and challenge them to find a strategy that will work. Draw their attention to the comment in the speech bubble.

Alternatively, you could discuss the strategy suggested by the speech bubble and ask *"How will this idea help us compare the proportions?"* The strategy involves converting both proportions to unit fractions (fractions with a numerator of 1) and then using the denominator to compare their sizes.

In Terry's recipe in question **1**, the proportion of tablespoons of powder to millilitres of milk is 4:250, which can be written as $\frac{4}{250}$ or $\frac{1}{62.5}$. Your students may need help with this last step. In Tracey's recipe, the proportion is 6:300 or $\frac{1}{50}$. Once both proportions are expressed as unit fractions, students should see that an acceptable ratio is somewhere between them. There are many possibilities, but one that is somewhere near the middle of the two unit fractions would best fit the context, for example, $\frac{1}{55}$. This would equate to a drink made with 4 tbs of powder and 4 x 55 = 220 mL of milk, or 5 tbs and 5 x 55 = 275 mL of milk, or 6 tbs and 6 x 55 = 330 mL of milk.

Students can use the unit fraction strategy again for question **2a**. They should find that a suitable recipe for Tiana will have a cordial:water ratio of between 1:5 and 1 : 5.5. This number line shows the range of potential values for the denominator:



A suitable mix for Tiana would be 1 : 5.2, 1 : 5.25, or 1 : 5.3. The middle ratio would translate into a drink made from 2 parts of cordial and 10.5 parts of water.

The challenge in question **2b** is to find a suitable unit against which the two recipes can be compared; one that doesn't involve working with unfriendly fractions. The best is 1 full glass. Here is a possible line of reasoning based on this unit: "Terry uses 6 tsp for three-quarters of a glass, so each quarter has 2 spoons. This means she would need 8 spoons for a whole glass. Tracey would use 10 spoons for 1 whole glass because 2 times 5 is 10. So Tracey's mix is stronger."

The most obvious ratio for a drink that has a "teaspoons of powder : glass of milk" ratio that lies between 8:1 and 10:1 is 9:1. This translates into 9 tsp of powder in a full glass of milk.

Page 9: Fully Grown

Achievement Objectives

- solve practical problems which require finding fractions of whole number and decimal amounts (Number, level 3)
- express quantities as fractions or percentages of a whole (Number, level 4)
- find a given fraction or percentage of a quantity (Number, level 4)

AC
EA
AA
AM
AP

Number Framework Links

Use this activity to:

- develop confidence in students who are beginning to use advanced proportional strategies (stage 8)
- help students consolidate and apply their knowledge of percentages (stage 8).

Activity

This context is usually of great interest to students. It is important to point out that the chart is about *likely* height and that, in reality, an individual's fully grown height can be quite different from what is predicted here.

Students should try to answer questions **1** and **2** without a calculator. A calculator could be used for questions **3** and **4**.

Have your students work in pairs. Ask them to examine David's strategy and try to explain to their classmate the reason for each step. Get them to write down the reasons, like this:

What David did	Why he did it
Changed 55% to $\frac{55}{100}$	He likes to think of percentages as fractions.
$\frac{55}{100} = \frac{11}{20}$	He simplifies the fraction.
99 ÷ 11 = 9 cm	Finding and knowing $\frac{1}{20}$ will make it easy to find $\frac{20}{20}$.
20 x 9 = 180 cm	This multiplication finds the full height ($\frac{20}{20}$) in cm.

Challenge your students to solve question **1** using David's strategy; it reinforces important ideas about percentages:

What I did	Why I did it
Changed 80% to $\frac{80}{100}$	I'm thinking of the percentage as a fraction.
$\frac{80}{100} = \frac{4}{5}$	I'm simplifying the fraction.
144 ÷ 4 = 36 cm	Finding and knowing $\frac{1}{5}$ will make it easy to find $\frac{5}{5}$.
36 x 5 = 180 cm	This multiplication finds the full height ($\frac{5}{5}$) in cm.

Another, more familiar strategy is a double number line. Students can enter the information they have and then extend the pattern to find the information they need:

Height (cm)	0	36	72		144	?
Percentage of full height	0	20	40	60	80	100

Question **2** is a simple step from question **1**. Now that David's likely adult height has been determined, it should be halved to get an estimate of what his height would have been at age 2.

Question **3** also depends on the students working from a correct answer to question **1**. If students are using calculators, get them to enter 85% as 0.85 instead of using the percentage key. Many calculators do not have percentage keys (because they are completely unnecessary), and students who use them are unlikely to understand what is going on mathematically.

Each of the calculations in question **3** can be done without a calculator, and you could challenge your students to do this, either before or following solution by calculator. Here is a possible line of reasoning for the first part: "We know from question **1** that David is currently 144 cm tall, which is 80% of his likely adult height. From our answer to the same question, we know that his likely adult height is 180 cm. 10% of 180 is 18 cm, which means that 5% of 180 is 9 cm.
144 + 9 = 153 cm."

Question **4** is challenging, so focus on finding an equation that will lead to the solution. The students can then use a calculator to solve the problem. Angie's adult height is likely to be 178 cm (2 cm shorter than her brother). She is David's twin, so her age must now be 10. According to the chart, her present height is 86% of her likely adult height. So the equation is:
$0.86 \times 178 = \square$. When they have solved the equation, the students need to remember to compare their answer with 144 cm (David's present height) to see what the difference in height should be.

Pages 10-11: Ratio Rip

Achievement Objectives

- find fractions equivalent to one given (Number, level 4)
- express quantities as fractions or percentages of a whole (Number, level 4)

AC
EA
AA
AM
AP

Number Framework Links

Use this activity to help students consolidate and apply their knowledge of equivalent fractions (stage 8).

Game

In this game, students practise recognising the relationship between the ratio of the parts *in* a whole and the fraction of the parts *to* the whole. They must know that a ratio of 2:5 (for example) within a set means that the two kinds of cube are present in the whole in the proportions $\frac{5}{7}$ and $\frac{2}{7}$. This principle is clearly illustrated by the diagram on page 11 in the students' book.

Get four players to demonstrate the game while the rest of the class observes. Designate a place for the stacks of cubes and label each stack with a card showing the ratio (for example, 1 red : 3 blue or 2 red : 5 blue). These cards will help players keep the stacks organised.

Ensure that your students recognise that they must join the same-coloured parts of two ratios, regardless of which coloured part is named first in each: 2 red to 5 blue is the same as 5 blue to 2 red.

After students have used the blocks to help them combine two ratios, they should separate them into their original stacks, ready to be used again. They are likely to find before long that they can image the operations and do not need to handle the blocks at all.

Emphasise the fourth rule. Players who are playing strategically will choose ratios that can't be matched by their opponents.

Once players are familiar with the rules, they can use this game for independent practice in small groups.

Later, students may like to experiment with changes to the rules and come up with their own variations on the game. For example, a rule change could say "If a player forms a ratio that is already covered on an opponent's grid, they can remove the covering counter." This variation may need a time limit, in which case, the winner would be the person with most fractions covered when the agreed time is up.

Pages 12-13: Laser Blazer

Achievement Objectives

- solve practical problems which require finding fractions of whole number and decimal amounts (Number, level 3)
- find a given fraction or percentage of a quantity (Number, level 4)
- find fractions equivalent to one given (Number, level 4)

AC
EA
AA
AM
AP

Number Framework Links

Use this activity to:

- develop confidence in students who are beginning to use advanced proportional strategies (stage 8)
- help students to consolidate and apply their knowledge of percentages (stage 8).

Activity

This activity provides an interesting context in which students find fractions and percentages and practise expressing fractions as percentages. The score printout is an excellent example of a double number line.

Use question **1** to introduce the context and to check students' understanding of percentages. Have the students use the think, pair, share technique to discuss possible strategies. A strategy might go like this: "Since 3 people can enter for the price of 2, it will cost $30 for 3, which is $10 per person." Or this: "If they all paid $15, it would be 15 x 6 = $90, but take off $30 for the two who don't have to pay and the total is $60. $60 ÷ 6 = $10 per person."

The total discount was $30 off the normal price of $90; the per-person discount was $5 off the normal price of $15. This means the discount was $\frac{30}{90} = \frac{5}{15} = \frac{1}{3}$. As a percentage (fraction of 100), this is $33\frac{1}{3}$%.

Students who do not understand how to write fractions as percentages could use 100 beads on a string as an open number line. The string is labelled from 0 at one end to 1 at the other, and the 100 beads represent one whole or 100%. Students can find $\frac{1}{3}$ of the string and see that it is $33\frac{1}{3}$%. Another useful aid is material master 7–4, Percentage Strips (available at www.nzmaths.co.nz/numeracy/materialmasters.aspx), which shows a drawing of the 100-bead number line. Both these models can help students image a percentage as the answer to the question "This would be equivalent to how much of 100?"

Have the students explain how the score printout shows not only the number of hits but the relative success of the player (their accuracy). They should generalise this in a $\frac{\text{hits}}{\text{shots}}$ statement, showing the hits as a fraction of total shots.

The students should try to express their statement as an equation or formula. They can use letters to stand for the two variables, hits and shots: accuracy = $\frac{h}{s}$. Then they should try to adapt their formula so that it expresses accuracy as a percentage: accuracy = $\frac{h}{s} \times 100$. They can use their formula five times to solve question **2** for each of the friends.

As a percentage, Luke's accuracy = $\frac{30}{50}$ x 100. This is equivalent to $\frac{3}{5}$ of 100. $\frac{1}{5}$ of 100 is 20, so $\frac{3}{5}$ x 100 = 20 x 3 = 60. This means that 60% of Luke's shots are hits.

Students who need the support of materials could use the 100-bead string or material master 7–4 to help them solve the parts of question **2**.

Question **3** switches the focus from the hits to shots ratio to the *remaining* hits to shots ratio needed to meet the 75% benchmark. This is not an easy idea for students to get their heads around, but the introductory diagram clearly shows them where to look for the data, and the thought bubble models the required thinking. For each printout, they need to work out:

- how many more hits the person needs to get to reach 53
- how many shots they have left (out of their total of 70).

Note that, in this case, students are asked to express their answers as fractions rather than percentages. The question has been constructed so that each answer, when simplified, is an everyday fraction. Once your students have the fractions, you could ask them to go a step further and express them also as percentages.

Question **3b** requires students to reflect on the meaning of their answers.

Pages 14-15: The Percentage Game

Achievement Objectives

- find fractions equivalent to one given (Number, level 4)
- express quantities as fractions or percentages of a whole (Number, level 4)

AC
EA
AA
AM
AP

Number Framework Links

Use this activity to:

- help students consolidate and apply their knowledge of percentages (stage 8)
- develop confidence in students who are beginning to use advanced proportional strategies (stage 8).

Game

This game is different in two ways. Firstly, the percentage questions are not all framed in the traditional way. Secondly, the winning strategy is not related to the speed that players travel round the board. As the Number Framework links above suggest, the game is for students needing skills reinforcement; *it is not intended as an introduction to percentages.*

Students will be used to statements where the unknown (the missing element) comes last, for example: 30% of 90 = $\square$, but the questions on the game board have the unknown at the beginning or in the middle. In addition to this, while each question could be written as an equation, it isn't. For these reasons, before introducing the game take some time to give students practice at rewriting questions as equations in their different formats and solving them.

Ask your students to consider and compare each of these equations or statements:

i. 30% of 90 = $\square$
ii. 30% of $\square$ = 90
iii. 27 is $\square$% of 90
iv. $\square$ is 30% of 90

i and iv are effectively the same: both are asking for the same part (30%) of the same whole (90). But i is expressed as an equation (a statement with an equals sign) while iv is not. Ask your students to identify the part of iv that is the equivalent of the equals sign. (It's the word "is".) iii asks what percentage one number (27) is of another (90). In ii, 90 is 30% of a number. The number must be greater than 90. The problem is to find what it is.

The player answering a question must do so without a calculator. But a competitor can use a calculator to check and challenge another player's answer. For this reason, it would be a good idea to have your students practise using the correct sequence of keystrokes. Encourage them to enter 30% as 0.3, 25% as 0.25 and so on, instead of $\frac{30}{100}$ or $\frac{25}{100}$ or by means of the percentage key. This will reinforce the fact that the percentage and its decimal equivalent are exactly the same mathematically, will ensure that students can work with percentages on a calculator that doesn't have a percentage key, and will often save completely unnecessary keystrokes.

Introduce the game by getting two or three students to play while the others watch. Make sure that everyone understands the rules and how the Choose option works. Model the answering and checking process. By checking with a calculator, the other players are also reinforcing useful skills. Make sure that everyone knows what they are meant to do with the 4 x 4 grid.

Students should dispose of any pencil and paper calculations they do so that they have to work them out afresh each time they play the game.

Once the players understand the way the game works, they can think about game strategy. Those who are able to work out the answers in their head will have an advantage because they can work out which squares they need to land on to complete their 3-in-a-rows.

The variation suggested has the potential to prolong the useful life of the game at the same time as it offers an excellent opportunity for students to come up with their own percentage questions, modelled on the patterns they have been given. New questions should be peer-reviewed before they get into circulation to ensure that the calculations don't involve tricky fractions or decimals.

Note that the variation uses the 5 x 5 grid. A winning line of 4 counters can be created using only numbered circles, or only blank circles, or a combination of numbered and blank circles.

Page 16: A Sizeable Problem

Achievement Objectives

- find a given fraction or percentage of a quantity (Number, level 4)
- express quantities as fractions or percentages of a whole (Number, level 4)

AC
EA
AA
AM
AP

Number Framework Links

Use this activity to:

- help students consolidate and apply their knowledge of percentages (stage 8)
- develop confidence in students who are beginning to use advanced proportional strategies (stage 8).

Activity

This imaginative scenario illustrates an important but often neglected principle related to increase and decrease, that is, if you increase an amount and then decrease it by the same percentage, you don't get back to the original amount. The same principle is at work every day in GST calculations: add $\frac{1}{8}$ (12.5%) or multiply by $\frac{9}{8}$ to include GST; subtract $\frac{1}{9}$ (11.1%) or multiply by $\frac{8}{9}$ to remove GST.

Check that your students can express the percentages on the buttons as their fraction or decimal equivalents. They may need to be told that the 67% and 133% buttons are rounded percentages but represent the fractions $\frac{2}{3}$ and $\frac{4}{3}$ respectively.

Discuss how an amount or size is changed when it is multiplied by a percentage less than 100% and more than 100%. The effect is exactly the same as when multiplied by a fraction less than 1 or more than 1.

Question **1** is carefully designed to clarify the principle discussed in the introductory paragraph above. Put the students into mixed-ability problem-solving groups (no more than four in a group). Get them to work through the question together and be prepared to discuss their thinking in the wider group.

Some students may find it difficult to get their head around question **1**: they are not told what Askaz's height is, so how can they increase it by 50%? If this is a problem, suggest that they make Askaz 100 cm high. (Any height will do, but 100 is easy to work with.) Demonstrate this height with the help of a metre rule or tape measure, and then increase it by 50% to 150 cm. Then reduce her new height by 50% and show that it is 75 cm, not the original 100 cm.

Ask *"What fraction of the new height is the original height?"* This can be written: $\frac{\text{original height}}{\text{new height}} = \frac{100}{150} = \frac{2}{3}$. So the students need to multiply 150 by $\frac{2}{3}$ to get back to the original height of 100 cm.

Now bring together the ratio for increase and the ratio for decrease so that the students can see how they are related: *"To increase the original height by 50%, we multiply it by $\frac{150}{100}$ or $\frac{3}{2}$. To get back to the original height, we multiply the new height by $\frac{100}{150}$ or $\frac{2}{3}$."* Height $\times \frac{3}{2} \times \frac{2}{3}$ = height.

Have the students examine this multiplication and see that $\frac{3}{2} \times \frac{2}{3} = \frac{6}{6} = 1$.

Before your students try question **2**, they should have a clear understanding of whether the fraction they are looking for will be greater or less than 1 and should understand that these fractions can equally well be expressed as percentages greater or less than 100%. A fraction and its percentage are mathematically identical: they only look different.

Question **3** brings together the ideas developed earlier. Students should now be able to explain why multiplying by the inverse fraction returns a size or an amount to its original state: "Multiplying by a pair of inverse fractions is the same as multiplying by 1. Multiplying by 1 leaves what we started with unchanged. Each of the inverses 'undoes' or 'neutralises' the effect of the other."

Two fractions that multiply together to make 1 are often called reciprocals. Reciprocals always come in pairs. It doesn't matter which is named first; each is the reciprocal of the other: $\frac{3}{4}$ and $\frac{4}{3}$, $\frac{5}{2}$ and $\frac{2}{5}$, $\frac{1}{10}$ and 10 ($\frac{10}{1}$), $\frac{7}{6}$ and $\frac{6}{7}$, 125% ($\frac{5}{4}$) and 80% ($\frac{4}{5}$).

You could ask your students why there is no button with 100% on it. 100% is the percentage equivalent of 1. (100% = $\frac{100}{100}$ = 1) There is no need for a 100% button because multiplying by 100% or 1 has no effect whatsoever.

Page 17: Pay Rates

Achievement Objectives

- write and solve problems which involve whole numbers and decimals and which require a choice of one or more of the four arithmetic operations (Number, level 3)
- find a given fraction or percentage of a quantity (Number, level 4)

AC
EA
AA
AM
AP

Number Framework Links

Use this activity to:

- encourage the transition from advanced multiplicative strategies (stage 7) to advanced proportional strategies (stage 8)
- help students consolidate and apply their knowledge of percentages (stage 8)
- develop confidence in students who are beginning to use advanced proportional strategies (stage 8).

Activity One

In this activity, students meet rates in a context of time worked and pay. Use problem-solving groups of not more than four to sort out the information needed for question **1** and then get the students to discuss and share strategies that might lead to a solution. Check that they have found the correct answers to question **1** because they are needed in the questions that follow.

If the students plan to use a trial-and-error strategy to solve question **2**, get them to make a sensible estimate first; this will encourage the more thoughtful approach described as trial and improvement. A simple estimate can be made by noting that the girls both earn around $5 an hour. This means that Mrs White must be paying out at the rate of about $10 an hour. If this is the case, both must work for about 20 hours (10 x 20 = $200).

If the students now take the estimate of 20 hours as a starting point, they find that $5 x 20 = $100, $4.50 x 20 = $90, and $100 + $90 = $190. This leaves $10, so the girls must have worked 1 more hour: 20 + 1 = 21 hours.

Alternatively, the students could combine the girls' pay and see that Mrs White is paying out a total of $9.50 per hour. $200 \div 9.5 = 200 \div 9\frac{1}{2} = 400 \div 19$. A strategy for dividing 400 by 19 might go like this: 190 ÷ 19 = 10, and another 190 ÷ 19 = 10. That means 380 ÷ 19 = 20, which leaves 20 ÷ 19 = 1 and a little bit. So the girls must both work 21 hours. It's a short step from here to the answer for question **2b**.

Activity Two

In question **1**, the girls' pay rates are adjusted. For Karen, the increase is 10%. This can be thought of as an addition: 10% of $5.00 is 50c; adding 50c to $5.00 gives $5.50 per hour. But encourage your students to think multiplicatively when making percentage increases or decreases and to express the mathematics in one-step equations. In Karen's case, the equation is: $5 x 1.10 = $5.50. In Kylie's case, the equation is: $4.50 x 1.20 = 4.50 x 1 + 4.50 x 0.20 = 4.50 + 0.90 = $5.40. Students can return to the methods they used in **Activity One** to find the answers to question **2**.

Question **3** extends students' thinking and should appeal to their sense of justice. Students need to see that Karen's pay should be $5.40, the same as Kylie's. They should also see that as a percentage, this increase must be less than the 10% that Karen was awarded. The increase from $5.00 is 40 cents. Here is one strategy for working out what this increase of 40 cents is as a percentage: "1% of $5.00 is 5c. 5c x 8 = 40c. So Karen's increase should have been 8% (8 x 1%)."

Have the students share their different ways of solving the question. Restate their explanations to clarify them and then record them, using numbers, on a chart or whiteboard.

Pages 18–19: The Equivalence Game

Achievement Objective

- find fractions equivalent to one given (Number, level 4)

AC
EA
AA
AM
AP

Number Framework Links

Use this activity to help students consolidate and apply their knowledge of equivalent fractions (stage 7).

Game

Once its structure and rules are understood, this adaptation of a well-known card game will be a popular choice as an independent practice and maintenance activity. By playing it, students will gain confidence with equivalent fractions and a sense for where they fit in fraction sequences.

Before the students begin playing, check that they have a strategy for understanding and finding equivalent fractions.

If they need the support of materials, they could use fraction pieces or Fraction Strips (Numeracy Project material master 7–7, available on www.nzmaths.co.nz/numeracy/materialmasters.aspx). If they are using number properties, the strategy is to multiply or divide by a suitable equivalent of 1. For example, $\frac{1}{3} \times \frac{3}{3} = \frac{3}{9}$ or $\frac{4}{6} \div \frac{2}{2} = \frac{2}{3}$. (Students need to be comfortable with the fact that 1 has an infinite number of such equivalents.) Those who need to practise making equivalent fractions using number properties could use Equivalent Fractions (material master 8–9).

The concept of a "run" of fractions is critical to the game, and students should study the illustrations and be able to describe exactly what constitutes one. All fractions in a run must have the same denominator or an equivalent form with that denominator. The first numerator does not have to be 1, but the numerators must increase consecutively by 1.

In this game, the lowest score always wins. Two scoring methods are suggested. The second will create a further mathematical challenge and should appeal to able and competitive students. The penalty of an extra 3 points is important as an encouragement to students to check each other's scores.

Pages 20–21: The Right Gear

Achievement Objectives

- solve practical problems which require finding fractions of whole number and decimal amounts (Number, level 3)
- find a given fraction or percentage of a quantity (Number, level 4)
- find fractions equivalent to one given (Number, level 4)

AC
EA
AA
AM
AP

Number Framework Links

Use this activity to:

- help students consolidate and apply their knowledge of equivalent fractions (stage 7)
- develop confidence in students who are beginning to use advanced proportional strategies (stage 8).

Activity

This activity provides a great context for an exploration of ratio. All students will have had some experience of gears on bicycles. Many will also have come across gears in construction sets. Educational suppliers have sets of plastic gears for sale and, if possible, you should have one of these sets available in your classroom.

Some important understandings:

- Gear wheels are often described by the number of teeth they have (for example, a 24-tooth wheel).
- Gear wheels can be joined by a chain. The effect is exactly the same as if they were meshed (teeth interlocked). Gear wheels designed to fit a chain are often called sprockets.
- If two wheels are meshed, the rate at which one turns the other is determined by the number of teeth each has. This is often expressed as a ratio, for example: 1:3.
- If a wheel with 12 teeth is driving a wheel with 36 teeth, the small wheel will have to turn exactly 3 times to make the big wheel turn once. And if the large wheel is driving the small one, the small wheel will turn exactly 3 times for every turn of the large one.
- If the ratio between the teeth of two wheels is 1:3 (as in the example above), the turns ratio will be 3:1.

Question **1** is designed to check whether students understand the basics or need help before moving on to question **2** and beyond.

In question **2**, they have to consider the combined effect of gear ratio and wind speed in a straightforward calculation that is intended to prepare them for the more complex problems that follow. Ask the students to explain how they will use the information. Encourage them to simplify the ratio to keep the calculation as easy as possible: 48:12 = 4:1, so 1 turn of the blades will turn the dynamo 4 times, and 20 turns of the blades will turn the dynamo 20 x 4 = 80 times.

Question **3** gives students practice with wind speed : gear ratio calculations of the kind they met in question **2**. The first line in the table has been completed as a model.

As an extra, you could show your students how multiplying pairs of numbers in the columns can reveal another equivalent relationship. They should find that wind speed x large wheel = small wheel x dynamo. This generalisation can be used to check each row. For example:
30 x 36 = 12 x $\square$, which means that 1 080 = 12 x $\square$, which means that $\square$ = 1 080 ÷ 12 = 90.

When introducing question **4**, ask the students *"What wind speed to dynamo ratio do we need for best performance?"* They should see that it is between 21:55 and 21:65. This is around 1:3, so they should identify combinations of wheels that are in the ratio 3:1, or very close. They are 36:12, 48:16, and 48:18.

If your students use a spreadsheet as suggested, they should examine the formula (=21*36/A3) and generalise the connections it makes: dynamo speed = wind speed x large gear ÷ small gear. This formula is applied to each row.

Challenge

This is a demanding problem. Students should spend time looking for a strategy that won't involve individually testing each combination of gears. Ask *"How can we use the wind speed range to help us find the range of gear wheels needed?"* A good reply would be: "If we can find which gears work for the slowest wind speed and which gears work for the highest wind speed, we should have the combination that will work best for the in-between wind speeds."

For a wind speed of 25, we need a ratio of between 1 : 2.2 (for a dynamo speed of 55) and 1 : 2.6 (for a dynamo speed of 65). For a wind speed of 45, we need a ratio of between 1 : 1.2 and 1 : 1.5.

No 3-gear combination will work with the 48-tooth wheel because only one small gear (24-tooth) fits within the required range. The only small gear that will *not* work with the 36-tooth gear is the 12-tooth one (36:12 is 3:1), so the three other gears (16-, 18-, and 24-tooth) must be the cluster we are looking for. Calculations will confirm this.

An alternative method of solving this problem is to set up a spreadsheet, as suggested in the activity. Here is part of a suitable spreadsheet:

B3 =B2*36/A3

Workbook 1

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1		Using 36–tooth wheel													
2	**Small Gear**	25	26	27	28	29	30	31	32	33	34	35	36	37	38
3	12	75	78	81	84	87	90	93	96	99	102	105	108	111	114
4	16	56	59	61	63	65	68	70	72	74	77	79	81	83	86
5	18	50	52	54	56	58	60	62	64	66	68	70	72	74	76
6	24	38	39	41	42	44	45	47	48	50	51	53	54	56	57
7															

The heart of the spreadsheet is the formula used to calculate the speed of the dynamo. The screenshot shows the formula that has been entered in cell B3; similar formulae must be entered in B4, B5, and B6 and then filled along each row. Students may not be familiar with the meaning of A3. This is what is known as an absolute cell reference and tells the program to keep referring back to the number in cell A3. To make a cell reference absolute, highlight it in the formula bar and press F4 (PC) or Apple T (Macintosh). Cells with values between 55 and 65 have been highlighted automatically using the Conditional Formatting option, found under Format on the menu bar.

Pages 22–23: Family Likeness

Achievement Objectives

- find a rule to describe any member of a number sequence and express it in words (Algebra, level 4)
- use a rule to make predictions (Algebra, level 4)
- sketch and interpret graphs on whole number grids which represent simple everyday situations (Algebra, level 4)

AC
EA
AA
AM
AP

Number Framework Links

Use this activity to develop confidence in students who are beginning to use advanced proportional strategies (stage 8).

Activity

This activity gives meaning to the notion of a linear (straight line) relationship. It should help students to think of a straight line graph as made up of many points (an infinite number) and to understand that each of those points can be described as a pair of numbers (co-ordinates) that are connected according to some rule (the "family likeness").

When introducing this activity, get your students to think about its title as a metaphor for the learning outcome (found at the bottom of the page): just as we can often tell that people are related because they have similar characteristics, patterns in numerical data often point to connections. Write the words *finding connections* on the whiteboard to emphasise the thinking that this activity aims to encourage.

Once the students have plotted the sets of points in question **1**, ask them what the lines tell them. The key ideas are:

- If a straight line can be drawn through a set of points, the points are members of the same "family".
- All other points on the same line are also members of that family.
- Any points that are not on the line are not part of that family.

Students should also note that the line places the ordered pairs in ascending or descending order based on the number for the *x* (horizontal) axis. This can be a help in finding the rule that connects the two parts of each pair.

Once the students have drawn the lines in question **1**, they should find it easy to name other ordered pairs that belong to the same "families".

In question **2**, students have to try to spot the connection that uniquely links each pair of *x* and *y* values in a family. They may find it a help to make a table showing consecutive *x* values and then enter the matching *y* values from the graph. A completed table for equation iv would look like this:

x value	0	1	2	3	4	5	6	7	8	9	10	11	12
y value	**-3**	**-2**	**-1**	**0**	**1**	**2**	3	**4**	**5**	6	**7**	**8**	9

Be aware that if students do create tables, they may need to use negative numbers to complete them (as in the example above). But this won't do them any harm!

Make it clear that the aim in question **2** is to find a rule that connects each pair of *x* and *y* values, not an observation about just one of the values. (Not, for example, "the *y* value goes up 1 each time".) Give your students every opportunity to discover for themselves how to find the connections.

One way of teaching this idea is to take a single ordered pair and get the group to brainstorm possible connections. For example, possibilities for the ordered pair (9, 6) include:

- "The *x* value plus the *y* value makes 15." [+]
- "The *x* value is 3 more than the *y* value." [–]
- "The *y* value is 3 less than the *x* value." [–]
- "The *y* value is $\frac{2}{3}$ of the *x* value." [x]
- "The *x* value is $1\frac{1}{2}$ times the *y* value." [x]
- "Double the *x* value equals triple the *y* value." [x]

When brainstorming possibilities, students should consider each of the four operations: +, –, x, and ÷. Once they have done this, they should take a second ordered pair from the same family and see which of the possible (brainstormed) connections is true for that pair too. If students find that two of their rules work, this will be because both express the same connection in different ways. (In the bulleted list above, 2 and 3 are just different ways of saying the same thing, as are 4, 5, and 6.) Once they have what they think is the correct connection, they should test it on the third ordered pair.

Students should be aware that a connection may involve more than one mathematical operation. For example, the rule that connects the three ordered pairs {(2,3), (5,15), (6,19)} is "multiply the *x* value by 4 and then subtract 5".

As they find the rules that connect the *x* and *y* value for each set of ordered pairs, students should try to express them as equations involving *x* and *y*. Each rule is able to be expressed in different ways, but the differences are cosmetic, not mathematical.

In question **3**, students must image extensions to the lines on their graph. By doing this, they will understand that the graph on the page is not a complete picture at all; it is just a small part of a graph that extends outwards off the edge of the page in all directions.

Have the students do question **4** by themselves and then swap their sets of ordered pairs with a classmate. Each should then try and work out the rule that connects the ordered pairs.

By way of extension, you could ask your students to consider the graph they created for question **1** and explain what is going on when two lines cross: which of the two families of points does this particular point belong to? (Both) Get them to check their ideas mathematically, putting the values from that ordered pair into the rule for each line. Is it possible for an ordered pair to belong to even more than two families of points? (Yes) See if they can find such a pair.

Page 24: da Vinci's Ratio

Achievement Objectives

- solve practical problems which require finding fractions of whole number and decimal amounts (Number, level 3)
- express a fraction as a decimal, and vice versa (Number, level 4)

AC
EA
AA
AM
AP

Number Framework Links

Use this activity to:

- develop confidence in students who are beginning to use advanced proportional strategies (stage 8)
- help students consolidate and apply their knowledge of decimals (stage 8).

Investigation One

da Vinci's famous proposition provides an excellent subject for encouraging students to explore how the ratio of two measures might be used to explore a possible relationship between them.

Question **1** is a practical task. Have the students make and record their 10 measurements, rounding them to the nearest centimetre. They should then use a calculator to divide each person's arm span by their height, entering the result (rounded to two decimal places) in the last column of the table. Before they go on to answer the questions, they should see if they can interpret the numbers they have found.

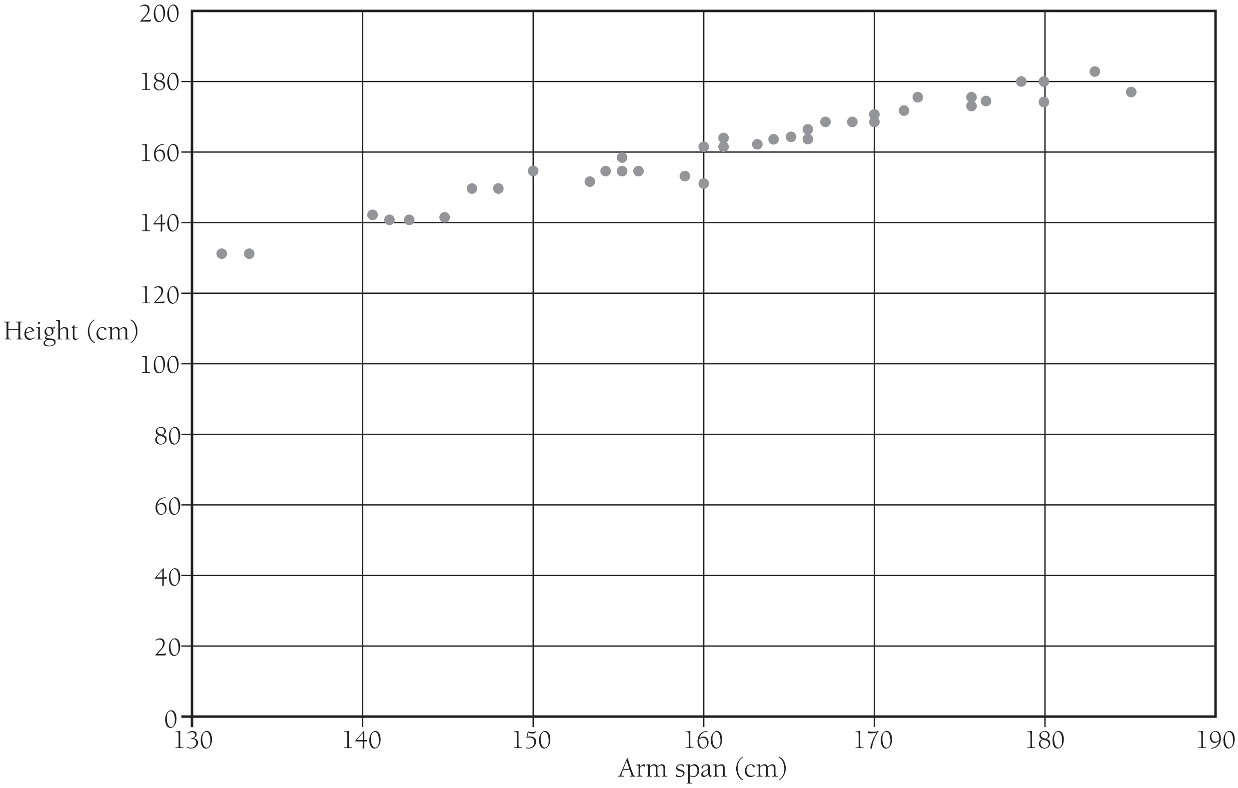
Question **2a** asks students to think about the significance of a ratio of 1. The only way we can divide one number by another and get 1 is if both numbers are the same, so a 1 in the ratio column means identical arm span and height measurements for that person, supporting da Vinci's hypothesis. A number greater than 1 means that the person's arm span is greater than their height; a number less than 1 means that the person's height is greater than their arm span. The closer the ratio is to 1, the better that person's measurements support da Vinci's hypothesis.

Some students may not be sure why a decimal number like 1.03 or 0.98 is being called a ratio when there is no colon. Ratios are often expressed as a single number, and like all ratios, these need to be understood in context. If a person's arm span is 146 cm and their height is 150 cm, this can be expressed as an arm span : height ratio of 146:150 or $\frac{146}{150}$ or just 0.97. If it helps, students can think of this as an arm span : height ratio of 0.97 : 1.

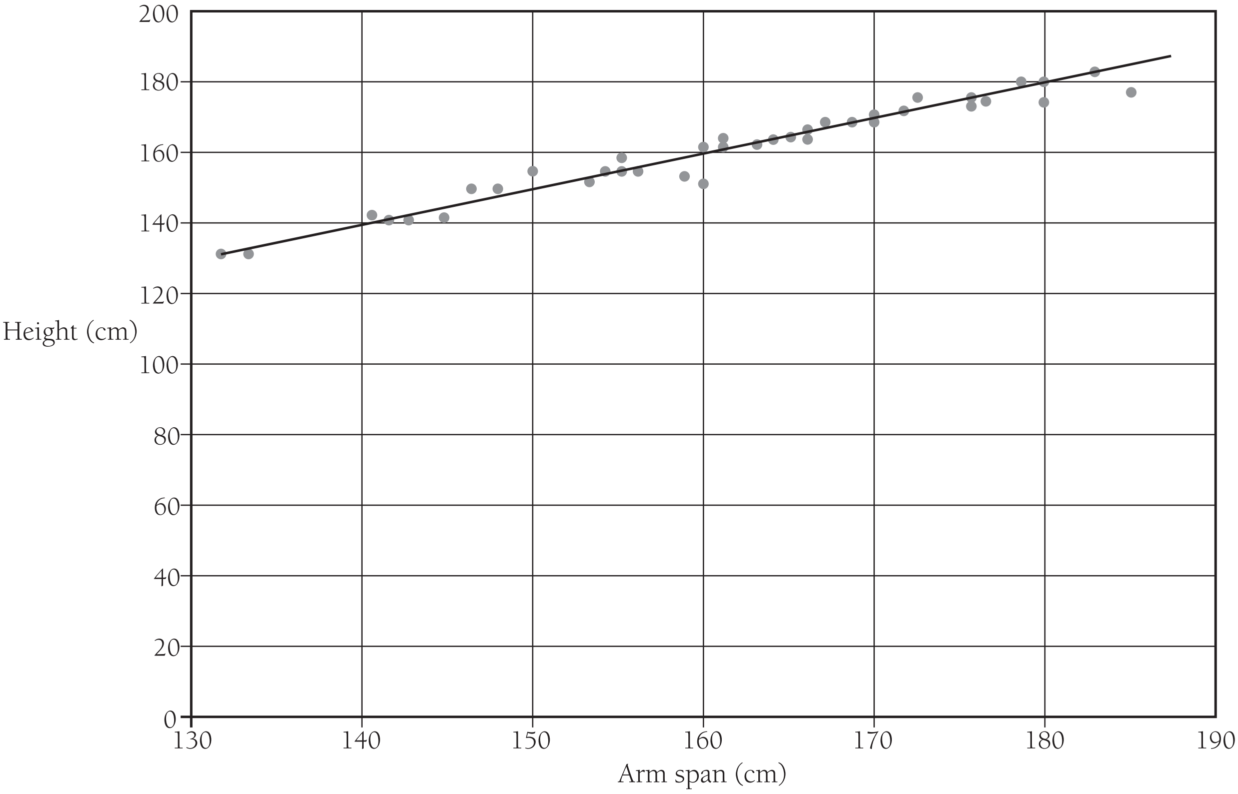
When they come to questions **2b–c**, your students should find that the ratios they have obtained strongly support da Vinci's hypothesis. Almost certainly, however, they will have one or two ratios that are a bit further from his ideal of 1. These can be explained as "natural variation".

There is an excellent activity based on da Vinci's observation at the New Zealand Census at School website: www.censusatschool.org.nz/resources/bank/masterpiece/ Students can download data from this site and explore it for themselves. A wealth of further data can be found on the United Kingdom Census at School website. See www.censusatschool.ntu.ac.uk

Question **2d** asks students to enter the measurements they have collected into a spreadsheet and create a scatter plot. Scatter plots are easy to create and are an excellent tool for investigating a possible relationship between two sets of data: if the data points appear to be grouped according to some pattern or principle, there is likely to be a relationship. Here is a scatter plot that uses data from the New Zealand Census at School website: the arm spans and heights of 38 level 3–4 students. Each point represents a single student:



It should be clear from this graph that (as da Vinci concluded) there is a remarkably close connection (correlation) between the measurements of arm span and height. In fact, the arm span : height ratio varies only between 0.97 and 1.04. If students use the Insert Trendline function to generate the line of best fit (see below), the correlation is even more obvious. Note that the line goes through (140,140), (160,160), and (180,180):



Students may like to investigate da Vinci's ratio further. If they do so, they will find that it is remarkably constant regardless of country, ethnic group, gender, or historical period. A different and very interesting investigation could focus on da Vinci the man.

Investigation Two

Unlike **Investigation One**, this investigation starts without an hypothesis, but you could challenge your students to come up with one and then try and show that it is or is not true.

A good starting point would be to get them to discuss in groups whether they think it likely that there is a relationship between a person's height and the circumference of their head and to explain why they think this.

Investigation should show that the height : head circumference ratio is very close to 3:1.

Copymaster: The Equivalence Game

1	2	3
4	6	8
9	12	16
24	1	2
3	4	6
8	9	12
16	24	

To play the game, four sets of this copymaster are needed (80 cards in all). If possible, enlarge by 120%. Cut along the dotted lines.

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