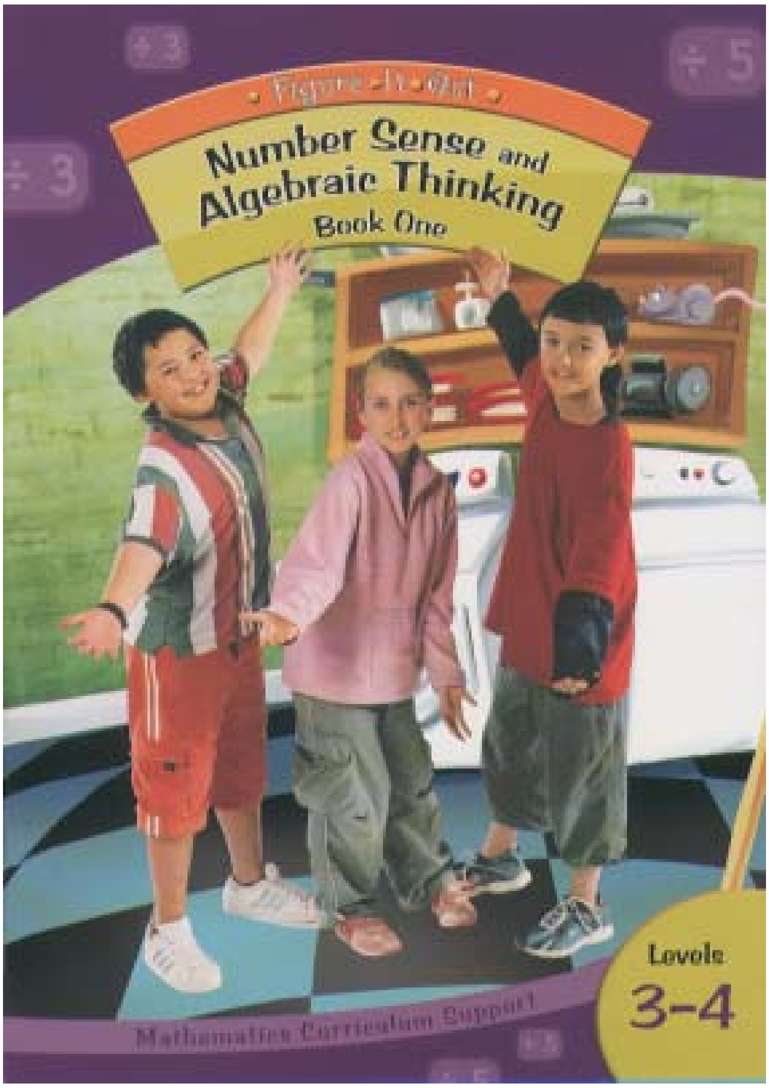
Answers and Teachers' Notes



MINISTRY OF EDUCATION
Te Tāhuhu o te Mātauranga

Contents

Introduction

The books for levels 3–4 in the Figure It Out series are issued by the Ministry of Education to provide support material for use in New Zealand classrooms. These books are most suitable for students in year 6, but you should use your judgment as to whether to use the books with older or younger students who are also working at levels 3–4.

Student books
The activities in the student books are set in meaningful contexts, including real-life and imaginary scenarios. The books have been written for New Zealand students, and the contexts reflect their ethnic and cultural diversity and the life experiences that are meaningful to students in year 6.

The activities can be used as the focus for teacher-led lessons, for students working in groups, or for independent activities. Also, the activities can be used to fill knowledge gaps (hot spots), to reinforce knowledge that has just been taught, to help students develop mental strategies, or to provide further opportunities for students moving between strategy stages of the Number Framework.

Answers and Teachers' Notes
The Answers section of the *Answers and Teachers' Notes* that accompany each of the student books includes full answers and explanatory notes. Students can use this section for self-marking, or you can use it for teacher-directed marking. The teachers' notes for each activity, game, or investigation include relevant achievement objectives, Number Framework links, comments on mathematical ideas, processes, and principles, and suggestions on teaching approaches. The *Answers and Teachers' Notes* are also available on Te Kete Ipurangi (TKI) at www.tki.org.nz/r/maths/curriculum/figure

Using Figure It Out in the classroom
Where applicable, each page starts with a list of equipment that the students will need in order to do the activities. Encourage the students to be responsible for collecting the equipment they need and returning it at the end of the session.

Many of the activities suggest different ways of recording the solution to the problem. Encourage your students to write down as much as they can about how they did investigations or found solutions, including drawing diagrams. Discussion and oral presentation of answers is encouraged in many activities, and you may wish to ask the students to do this even where the suggested instruction is to write down the answer.

> The ability to communicate findings and explanations, and the ability to work satisfactorily in team projects, have also been highlighted as important outcomes for education. Mathematics education provides many opportunities for students to develop communication skills and to participate in collaborative problem-solving situations.
>
> *Mathematics in the New Zealand Curriculum,* page 7

Students will have various ways of solving problems or presenting the process they have used and the solution. You should acknowledge successful ways of solving questions or problems, and where more effective or efficient processes can be used, encourage the students to consider other ways of solving a particular problem.

Figure It Out

Number Sense and Algebraic Thinking
Levels 3–4: Book One

Answers

Page 1: Speaking in Riddles

Activity

1. 2, 2, and 9
2. Strategies may vary, but you need to systematically list the possibilities and constraints in the problem to find the answer that works. To make sure there are no other possible answers, you need to try all the sets of 3 factors that multiply to 36. (For example, 3, 3, and 4 multiply to 36, but they do not add to an odd number.) You could use a table, such as:

Do the numbers:	multiply to give 36?	add to an odd number?	include two the same?	Is the pair less than the third number?
1, 1, 36	✓	x		
1, 2, 18	✓	✓	x	
1, 3, 12	✓	x		
1, 4, 9	✓	x		
1, 6, 6	✓	✓	✓	x
2, 2, 9	✓	✓	✓	✓
2, 3, 6	✓	✓	x	
3, 3, 4	✓	x		
4, 4, ?	x			
5, 5, ?	x			

(Note that you eliminate a group of 3 numbers as soon as a x applies.)

3. Problems will vary.

Pages 2–3: The Great Factor Hunt

Activity One

1. =10/A3
2. The numbers in column A that match whole numbers in column B are the factors of 10, that is, 1, 2, 5, and 10.
3. a. 1, 2, 3, 4, 6, 12

   b. 1, 2, 4, 7, 14, 28

   c. 1, 17

   d. 1, 2, 3, 4, 6, 9, 12, 18, 36
4. a. 1, 7, 29, 203

   b. No. Explanations will vary. Factors are always whole numbers, and none of the numbers between 10 and 202 will divide evenly into 203. 203 ends in a 3, so if you were trying all the numbers, you could start by eliminating all even numbers and numbers ending in 5. A quicker way is to recognise that the only factor between 2 and 9 shown on Tino's spreadsheet is 7, which has 29 as its matching pair. 7 and 29 are both prime numbers, so you cannot double and halve, treble and third, and so on to get whole numbers that are factors of 203. For example, 7 x 2 = 14 and 29 ÷ 2 = 14.5

Activity Two

1. **a.** Every even number after 2 has 2 as a factor, so except for 2, even numbers can't be prime numbers.

   b. 101. (97 is the closest prime number less than 100, but 101 is closer to 100.)

   c. 1 isn't a prime number because it has only 1 factor, itself. All prime numbers have 2 factors.

2. **a.** Every factor has a matching "other factor". For the number 18, 1 is matched with 18, 2 with 9, and 3 with 6. So 18 has 6 factors.

   b. **i.** 1, 4, 9, 16, 36, 49, or the product of any number multiplied by itself

   ii. Square numbers

   iii. The square root of the number gives it a factor that has the same "other factor". For example, $\sqrt{64} = 8$. $8 \times 8 = 64$, so 8 is a factor of 64. The factors of 64 are 1, 2, 4, 8, 16, 32, and 64, a total of 7 factors, which is an odd number of factors.

3. 65. (The factors of 65 are 1, 5, 13, 65.)

4. 81. (The factors of 81 are 1, 3, 9, 27, 81.)

5. Problems will vary.

Pages 4–5: The Factoring Factory

Activity One

1. Shaun had 15 times more coins. (x 3, then x 5 gives x 15.)

2. **a.** 6 times

   b. 12 times

   c. 20 times

   d. 27 times

   e. 60 times

   f. 16 times

Activity Two

1. $\frac{1}{6}$. ($24 \div 2 = 12$, $12 \div 3 = 4$; $\frac{4}{24} = \frac{1}{6}$)

2. **a.** $\frac{1}{10}$. (÷ 2, then ÷ 5 gives ÷ 10)

   b. $\frac{1}{20}$. (÷ 4, then ÷ 5 gives ÷ 20)

   c. $\frac{1}{18}$. (÷ 6, then ÷ 3 gives ÷ 18)

   d. $\frac{1}{10}$. (÷ 5, then ÷ 2 gives ÷ 10)

   e. $\frac{1}{8}$. (÷ 2, then ÷ 2, then ÷ 2 gives ÷ 8)

   f. $\frac{1}{45}$. (÷ 3, then ÷ 3, then ÷ 5 gives ÷ 45)

3. **a.** **i.** x 3, then x 3, then x 3.
   ($4 \times 3 \times 3 \times 3 = 108$, and $27 \times 4 = 108$.)

   ii. ÷ 4 ÷ 4 or ÷ 2 ÷ 2 ÷ 2 ÷ 2

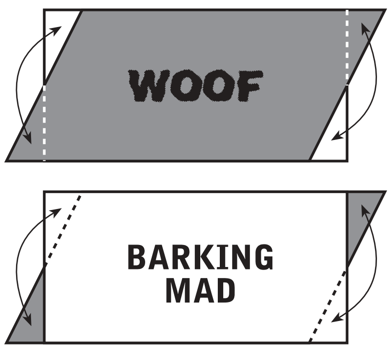
   b. For **a i**, there is only 1 combination that would work, but for **a ii**, there are 2 possibilities because ÷ 2 ÷ 2 has the same effect as ÷ 4.

4. No, only prime numbers are needed on the dials. The $\boxed{\times 4}$ and $\boxed{\div 4}$ dials are not needed because x 2 x 2 and ÷ 2 ÷ 2 have the same effect. The $\boxed{\times 6}$ and $\boxed{\div 6}$ dials are not needed because x 3 x 2 and ÷ 3 ÷ 2 have the same effect.

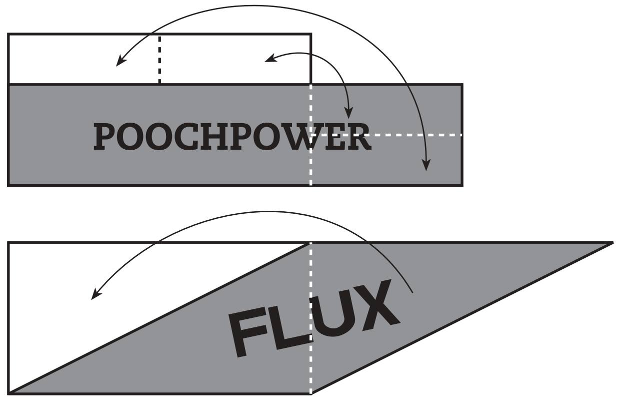
Pages 6–7: That Takes the Biscuit

Activity

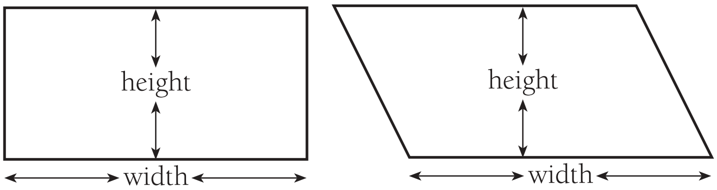
1. The two different-shaped biscuits are the same size because their areas match (assuming, based on Brock's comment, that they are the same thickness).



2. **a.–c.** All the biscuits have the same area.



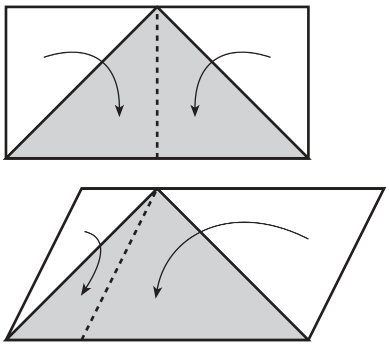
3. Answers will vary. If the widths and heights of the rectangles and parallelograms match, then the areas will be the same.



However, rectangles and parallelograms can have the same area even if these measurements don't match.

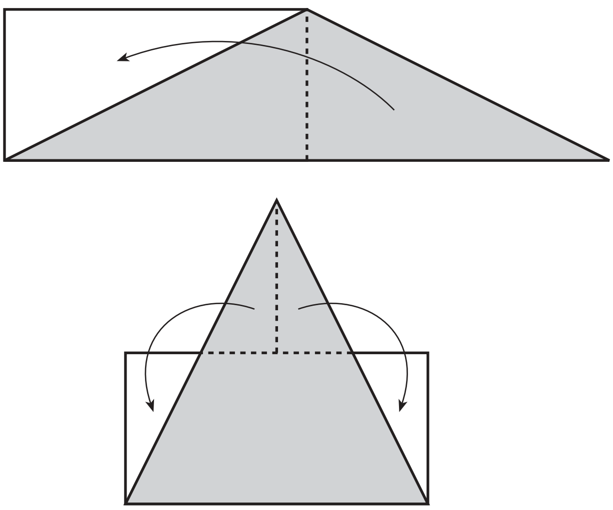
4. **a.** Half the area

   b. Explanations will vary. The diagrams below show the matching areas:



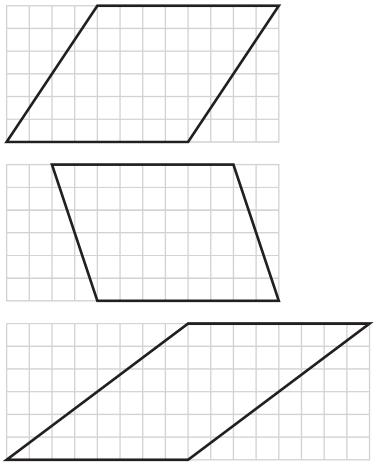
   c. With a rectangle. It is easier to see how a rectangle can be cut to make 2 triangles of the same size.

5. **a.–b.** Both biscuits are the same size as the Barking Mad biscuits. The diagrams below show the matching areas:

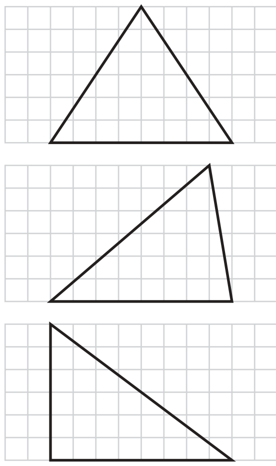


6. Shapes will vary. Some examples are:

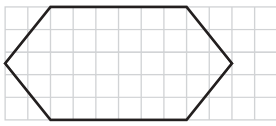
   a.



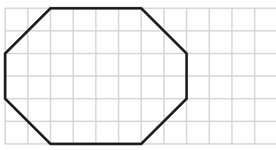
   b.



7. **a.** Hexagon examples will vary. One possible hexagon is:



   b. Rotation may vary, but the octagon with 40 squares will look like this:



Pages 8–9: Stripping Fractions

Activity

1. **a.** $\frac{2}{5}$. ($\frac{7}{5} = \frac{5}{5} + \frac{2}{5}$; $\frac{5}{5} = 1$, so $\frac{7}{5} = 1\frac{2}{5}$)

   b. Possible answers include $1\frac{2}{5}$, $\frac{1}{5} + \frac{6}{5}$, $\frac{2}{5} + \frac{5}{5}$, $\frac{3}{5} + \frac{4}{5}$

2. a. $\frac{2}{3} + \frac{2}{3} = \frac{4}{3}$
$= 1\frac{1}{3}$

b. $\frac{4}{7} + \frac{6}{7} = \frac{10}{7}$
$= 1\frac{3}{7}$

c. $\frac{5}{4} + \frac{3}{4} = \frac{8}{4}$
$= 2$

d. $\frac{7}{8} + \frac{5}{8} + \frac{3}{8} = \frac{15}{8}$
$= 1\frac{7}{8}$

3. You add the numerators and keep the denominator the same.

4. Yes. For subtraction of fractions that have the same denominator, you subtract the second numerator from the first and keep the denominator the same.

a. $\frac{4}{7}$. ($9 - 5 = 4$, so $\frac{9}{7} - \frac{5}{7} = \frac{4}{7}$)

b. $\frac{1}{3}$. ($3 - 2 = 1$, so $\frac{3}{3} - \frac{2}{3} = \frac{1}{3}$)

c. $\frac{4}{8}$ or $\frac{1}{2}$. ($7 - 3 = 4$, so $\frac{7}{8} - \frac{3}{8} = \frac{4}{8}$, which is $\frac{1}{2}$)

5. a. $1\frac{1}{4}$. ($\frac{1}{2} = \frac{2}{4}$; $\frac{3}{4} + \frac{2}{4} = \frac{5}{4}$ or $1\frac{1}{4}$)

b. $1\frac{1}{2}$. ($\frac{2}{3} = \frac{4}{6}$; $\frac{5}{6} + \frac{4}{6} = \frac{9}{6}$ or $1\frac{1}{2}$)

c. $1\frac{1}{8}$. ($\frac{3}{4} = \frac{6}{8}$; $\frac{6}{8} + \frac{3}{8} = \frac{9}{8}$ or $1\frac{1}{8}$)

d. $1\frac{3}{10}$. ($\frac{3}{5} = \frac{6}{10}$; $\frac{7}{10} + \frac{6}{10} = \frac{13}{10}$ or $1\frac{3}{10}$)

e. $1\frac{1}{4}$. ($\frac{1}{2} = \frac{2}{4}$; $\frac{7}{4} - \frac{2}{4} = \frac{5}{4}$ or $1\frac{1}{4}$)

f. $\frac{2}{3}$. ($\frac{4}{3} = \frac{8}{6}$; $\frac{8}{6} - \frac{4}{6} = \frac{4}{6}$ or $\frac{2}{3}$)

g. 1. ($\frac{2}{5} = \frac{4}{10}$; $\frac{14}{10} - \frac{4}{10} = \frac{10}{10}$ or 1.
Or: $\frac{14}{10} = \frac{7}{5}$; $\frac{7}{5} - \frac{2}{5} = \frac{5}{5}$ or 1)

h. $\frac{5}{8}$. ($\frac{3}{4} = \frac{6}{8}$, so $\frac{11}{8} - \frac{6}{8} = \frac{5}{8}$)

6. Using equivalent fractions, you rewrite the problem so that all the fractions have the same denominator. You then add or subtract the numerators as required.

Pages 10–11: What to Do?

Activity

1. 1 403 – 874 = $\square$. Felise needs 529 more stamps.

2. a. Diagram **i** shows Asosi's problem. He took out \$986 and had \$1,309 left, so his original amount was \$986 + \$1,309.

b. \$2,295. (986 + 1 309)

3. Possible equations and their answers are:
Number of plums = 7 x 39
7 x 39 = 273

or:
Number of plums = (7 x 40) – 7
(7 x 40) – 7 = 280 – 7
= 273

4. $\square$ x 24 = 864
or $\square$ = 864 ÷ 24
= 36 cartons

5. $\square$ = 52 ÷ 4
= 13 years. So they are now 12 years old.
A possible box diagram of this problem is:

52			

6. $\square$ = 756 ÷ 9
= 84 cards
A possible box diagram of this problem is:

Jackie: 756								
Guy								

7. 1 248 ÷ 3 = 416, so Sophie has 416, and Chris has 2 x 416 = 832.
A possible box diagram of this problem is:

1 248		
Sophie	Chris	Chris

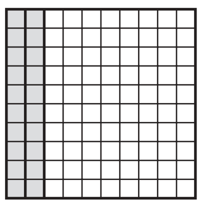
8. 9 in the front carriage, 12 in the middle carriage, and 17 in the last carriage.
One way to solve this is:
There are 38 passengers altogether.
Last = front + 8
Middle = front + 3 (or = front + 8 – 5)
Front = (38 – 8 – 3) ÷ 3
= 9
(So the last carriage has 9 + 8 = 17, and the middle carriage has 17 – 5 = 12 or 9 + 3 = 12.)

38				
9 (front)	3	9 (middle)	8	9 (last)

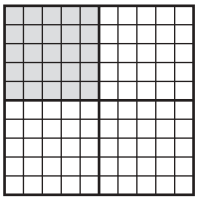
Pages 12–13: Pondering Percentages

Activity

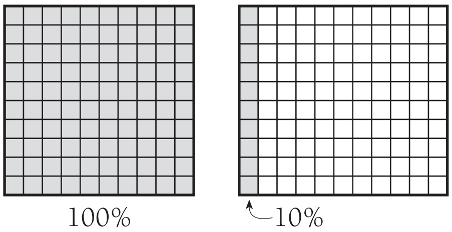
1. Yvette
$\frac{1}{5}$ is $\frac{2}{10}$. $\frac{1}{10}$ = 10%, so $\frac{1}{5}$ = 20%.



Leonie
$\frac{1}{2}$ = 50%, and $\frac{1}{4}$ is half of $\frac{1}{2}$, so $\frac{1}{2}$ of 50% must be 25%.



Andrew
Yes, you can. 110% would be $\frac{1}{10}$ more than 1.

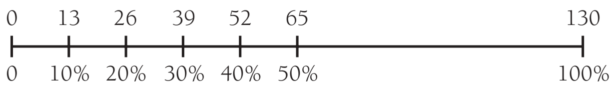


2. **a.** Answers and a possible method for each (based on one of the ideas shown) are:

   i. \$11. 10% or $\frac{1}{10}$ of 55 is 5.50.
   So 20% is 2 x 5.50 = \$11.00

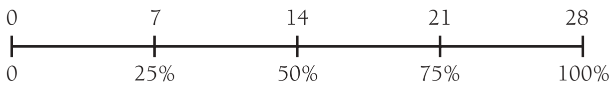
   ii. \$12. 10% of 80 is 8. 5% is 4.
   8 + 4 = \$12

   iii. \$52



   40% of 130 is \$52.

   iv. \$21



   75% of 28 is \$21.

   v. \$38. 95% is 100% – 5%. 10% of 40 is 4, so 5% is 2. 40 – 2 = \$38.

   vi. \$60. 100% of 48 is 48.
   25% of 48 is 12. 48 + 12 = \$60.

   b. Discussion and opinions will vary. (All the methods will work.)

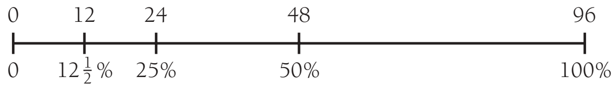
   c. Answers and a possible method for each (based on one of the ideas shown) are:

   i. \$14.40. 50% of 36 is 18;
   10% of 36 is 3.60; 18 – 3.60 = \$14.40

   ii. \$28. 10% of 35 is 3.50;
   3.50 x 8 = \$28
   or 35 – 3.50 – 3.50 = \$28

   iii. \$15.60. 50% of 24 is 12;
   10% of 24 is 2.40; 5% of 24 is 1.20;
   12.00 + 2.40 + 1.20 = \$15.60

   iv. \$12



Pages 14–15: Close Ties

Activity

1. **a.** Alice and Ruth are both right. Kimiora is wrong. The size of a fraction is based on both the numerator (top number) and the denominator (bottom number). The difference between these numbers by itself will not give the size of the fraction.

   b. Alice is wrong. 999 is a large number, but as a numerator, it's small in relation to the denominator, 10 000.
   999 is nearly $\frac{1}{10}$ of 10 000, so the fraction is close to $\frac{1}{10}$.
   Ruth is right, but it's not just the big number on the bottom that makes the fraction small. For example, $\frac{9\,999}{10\,000}$ has 10 000 on the bottom but is very close to 1.
   Kimiora is right. $\frac{5}{10}$ is $\frac{1}{2}$, so $\frac{1}{10}$ must be closer to 0.

2. Answers will vary. You can't tell if a fraction is close to 0, $\frac{1}{2}$, or 1 simply by looking at the numerator or the denominator. You need to look at the size of the numerator *compared with* the denominator. If necessary, you can round one or both of the numbers to make this easier. If the numerator is about half of the denominator, the fraction is closer to $\frac{1}{2}$, if it is a lot less than the denominator, it is closer to 0, and if it is nearly as big as the denominator, it is closer to 1. Ask: "What do you multiply the numerator by to get the denominator?"

3. **a.** Closer to 1

   b. Closer to $\frac{1}{2}$

   c. Closer to 0

   d. Closer to 1. (It's actually bigger than 1.)

4. You are right if you agree with Ruth's and Alice's thinking. Both should be able to get the correct answer.
Ruth: $\frac{7}{16}$ is smaller than $\frac{8}{16}$, which is $\frac{1}{2}$, so $\frac{1}{2} + \frac{7}{16} < 1$ (is less than 1).
Alice is right in changing $\frac{1}{2}$ to $\frac{8}{16}$ so she can add $\frac{8}{16} + \frac{7}{16} = \frac{15}{16}$. $\frac{15}{16} < 1$
Kimiora is wrong. You don't add the denominators, and the denominators have to be the same before you can add the numerators. You can't add a fraction bigger than $\frac{1}{18}$ (in this case, $\frac{7}{16}$) to $\frac{1}{2}$ and finish with an answer that is less than $\frac{1}{2}$.

5. The answer is equal to 1.
($\frac{2}{5} + \frac{6}{10} = \frac{4}{10} + \frac{6}{10}$
$= \frac{10}{10}$
$= 1$)

Pages 16–19: Fishy Fractions

Activity One

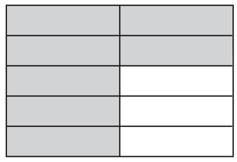
1. Any fraction between $\frac{1}{2}$ and $\frac{3}{4}$ is correct, but Kylie is probably looking for a fraction as close to halfway between $\frac{1}{2}$ and $\frac{3}{4}$ as possible. The shrimp biscuits are divided into twelfths. $\frac{1}{2}$ is $\frac{6}{12}$, and $\frac{3}{4}$ is $\frac{9}{12}$, so a "close to halfway" fraction is $\frac{7}{12}$ or $\frac{8}{12}$.

2. The fractions given below are close to halfway between the two fractions Kylie has tried in each scenario. They are based on twelfths because that is how the shrimp biscuits are shown on the page. Other fractions close to the halfway point are also acceptable.
   a. $\frac{1}{4}$ is $\frac{3}{12}$, and $\frac{1}{2}$ is $\frac{6}{12}$, so $\frac{4}{12}$ or $\frac{5}{12}$ should work.
   b. $\frac{2}{3}$ is $\frac{8}{12}$, and $\frac{1}{2}$ is $\frac{6}{12}$, so $\frac{7}{12}$ should work.
   c. $\frac{5}{6}$ is $\frac{10}{12}$ and $\frac{2}{3}$ is $\frac{8}{12}$, so $\frac{9}{12}$ should work. ($\frac{9}{12}$ is $\frac{3}{4}$.)
   d. $1\frac{1}{3}$ is $1\frac{4}{12}$, and $1\frac{1}{2}$ is $1\frac{6}{12}$, so $1\frac{5}{12}$ should work.

3. Strategies may vary. First, you have to divide the biscuit into parts so that each fraction can be shown. The strategy used above is to turn the fractions into twelfths and find the halfway point of the numerators (the top numbers). Another strategy is to halve the parts of the shrimp biscuit until you can find an "in-between" fraction.

Activity Two

1. a. The biscuit would need to be divided into tenths, not twelfths.



   b. $\frac{8\frac{1}{2}}{12}$ is the same as $\frac{17}{24}$.
   c. $\frac{1}{2}$ of $\frac{17}{12}$ is $\frac{8\frac{1}{2}}{12}$. (You need to divide the numerator by 2, but the denominator stays the same.) $\frac{8\frac{1}{2}}{12}$ is the same as $\frac{17}{24}$.
   d. Yes, $\frac{5}{7}$ is between $\frac{2}{3}$ and $\frac{3}{4}$. Using equivalent fractions, $\frac{2}{3} = \frac{56}{84}$, $\frac{5}{7} = \frac{60}{84}$, and $\frac{3}{4} = \frac{63}{84}$. (84 is the lowest common denominator for 3, 7, and 4.) 60 is between 56 and 63.

2. Answers may vary. Chris and Hannah have similar strategies because they both find the value halfway between the two fractions.

3. a.
   - Decimals method (Charu):
     $\frac{5}{4} = 1.25$, $\frac{6}{4} = 1.5$. Any fraction that converts to a decimal between 1.25 and 1.5 will do. For example, $1\frac{1}{3}$ ($1.\dot{3}$) or $1\frac{4}{10}$ (1.4).
   - Halfway method (Chris):
     $\frac{5\frac{1}{2}}{4}$, which is $\frac{11}{8}$ or $1\frac{3}{8}$
   - Averaging method (Hannah):
     $\frac{5}{4} + \frac{6}{4} = \frac{11}{4}$, $\frac{11}{4} \div 2 = \frac{5\frac{1}{2}}{4}$, which is $\frac{11}{8}$ or $1\frac{3}{8}$
   - Halfway between denominators and numerators method (Vaitoa):
     $\frac{5\frac{1}{2}}{4} = \frac{11}{8}$ or $1\frac{3}{8}$

   b.
   - Decimals method (Charu):
     $2\frac{3}{4} = 2.75$, $2\frac{7}{8} = 2.875$. So any decimal between 2.75 and 2.875 will do. For example, $2.8 = \frac{28}{10}$, which is $\frac{14}{5}$ or $2\frac{4}{5}$.
   - Halfway method (Chris):
     $2\frac{3}{4} = 2\frac{6}{8}$, so $2\frac{6\frac{1}{2}}{8}$ is midway between the two numbers, that is, $2\frac{13}{16}$.
   - Averaging method (Hannah):
     $2\frac{3}{4} + 2\frac{7}{8} = 2\frac{6}{8} + 2\frac{7}{8}$
     $= 4\frac{13}{8}$
     Half of $4\frac{13}{8}$ = 2 wholes and $\frac{6\frac{1}{2}}{8}$, that is, $2\frac{13}{16}$.

- Halfway between denominators and numerators method (Vaitoa): $2\frac{3}{4} = \frac{11}{4}$, $2\frac{7}{8} = \frac{23}{8}$. The average (mid value) of 11 and 23 is 17; the average of 4 and 8 is 6. $\frac{17}{6} = 2\frac{5}{6}$

Pages 20–21: Tile the Town, Tiny!

Activity

1. 10. (10 x 100 = 1 000)

2. Practical activity. Answers are:

a. 5 000

b. 3 600

3. a. 240

b. 4 000

c. 24 000

d. 21 000

4. The use of the strategies will vary.

i. 27 + 23 = 50. 50 ÷ 2 = 25.
25 x 25 = 625 cm^2; or 30 x 20 = 600 cm^2

ii. 45 + 65 = 120. 120 ÷ 2 = 60.
60 x 60 = 3 600 cm^2; rounding, such as 50 x 70 = 3 500 cm^2

iii. 50 x 40 = 2 000 cm^2

iv. 10 x 65 = 650 cm^2 or 15 x 60 = 900 cm^2

v. 34 + 46 = 80, 80 ÷ 2 = 40,
40 x 40 = 1 600 cm^2; or 30 x 50 = 1 500 cm^2

vi. 50 x 40 = 2 000 cm^2

vii. 190 x 20 = 3 800 cm^2 or
200 x 20 = 4 000 cm^2

viii. 400 x 300 = 120 000 cm^2

5. a. i. 621 cm^2

ii. 3 375 cm^2

iii. 1 862 cm^2

iv. 806 cm^2

v. 1 564 cm^2

vi. 1 862 cm^2

vii. 4 158 cm^2

viii. 120 768 cm^2

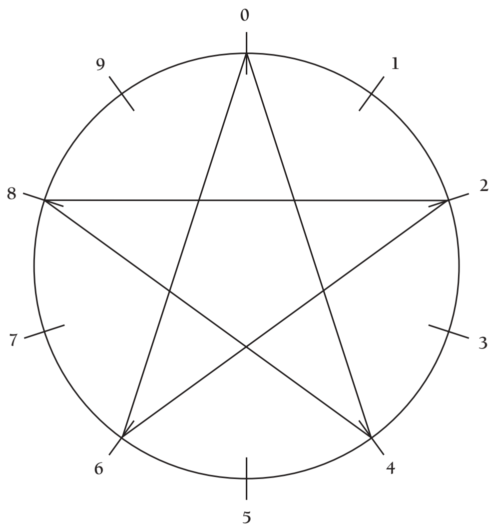
b. It depends on the numbers being rounded. If both numbers are close to decades, rounding both to the nearest decade will give a good estimate. Where the two numbers are roughly the same amount either side of a decade, rounding one up and the other down will give a good result.

c. The add, halve, and square method gives a good result if the two numbers are fairly close. The further apart the numbers are, the less satisfactory the estimate will be. This is because the method treats a rectangle as a square.

Pages 22–24: Wheeling and Dealing

Activity One

1. a. Practical activity. The pattern of the ones digits repeats as you carry on to higher multiples of 4. The plotting line for the pattern visits only even digits, 0, 2, 4, 6, 8, and forms the following star polygon.



b. No. 47, 95, and 783 are all odd numbers and therefore do not have 0, 2, 4, 6, or 8 as their ones digit.

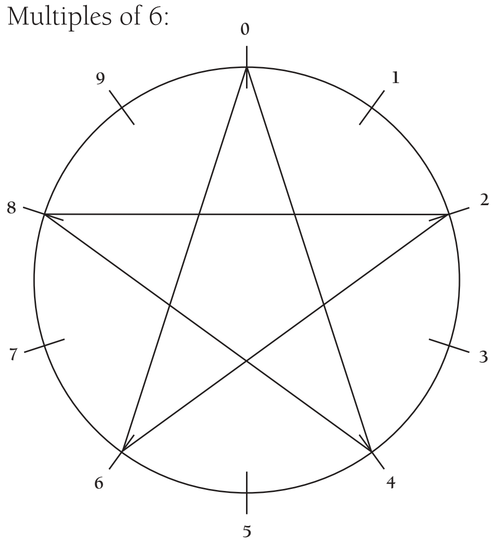
2. a. A vertical line joining 0 and 5

b. If the ones digit is 0 or 5, the number is a multiple of 5.

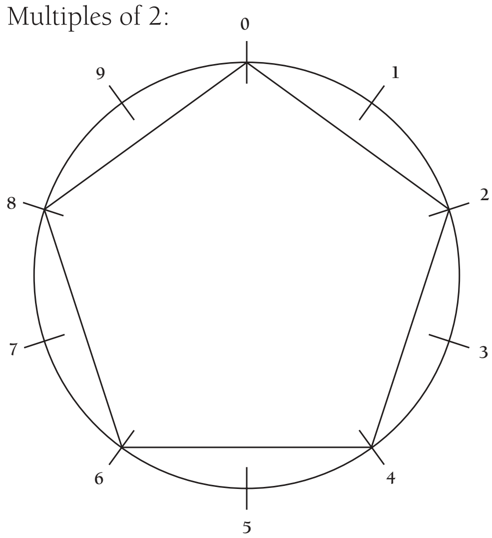
3. a. Multiples of 3 or 7 or any numbers with a ones digit of 3 or 7

b. Yes. The ones digits would form the same pattern because the tens digits do not affect the pattern.

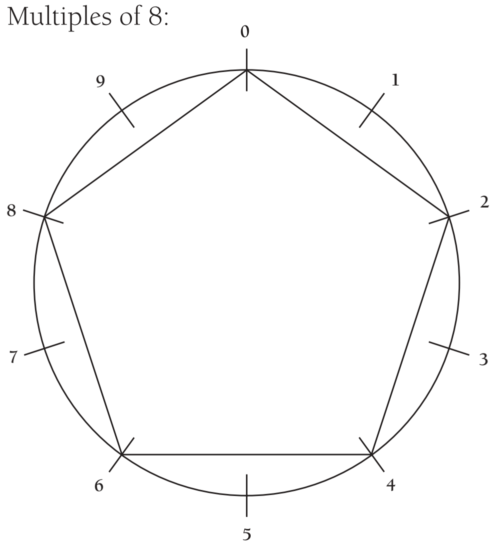
4. Practical activity
Multiples of 6:



Multiples of 2:



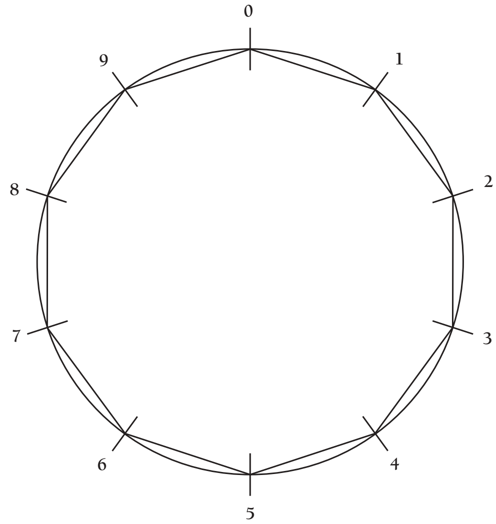
Multiples of 8:



The multiples of 6 form a star like the multiples of 4.
The multiples of 2 and 8 form a pentagonal pattern. The only difference is the order in which the points are joined.

5. **a.** Same as multiples of 1

b. Decagon, like multiples of 9
The following diagram applies to **a** and **b.**



c. Pentagon, like multiples of 8. (See the diagram for the multiples of 8 at question **4.**)

Activity Two

1. **a.** 4 + 2 = 6, so 42 is divisible by 3.

b. 7 + 5 = 12, so 75 is divisible by 3.

c. 8 + 8 = 16. 16 is not divisible by 3, so 88 is not divisible by 3.

d. 10 + 2 = 12, so 102 is divisible by 3.

2. **a.** 2 + 7 + 3 = 12, so 273 is divisible by 3.

b. 4 + 1 + 4 = 9, so 414 is divisible by 3.

c. 5 + 2 + 3 = 10. 10 is not divisible by 3, so 523 is not divisible by 3.

d. 6 + 7 + 2 = 15, so 672 is divisible by 3.

e. 1 + 1 + 1 + 0 = 3, so 1 110 is divisible by 3.

f. 3 + 5 + 6 + 1 = 15, so 3 561 is divisible by 3.

3. Yes, because multiples of 9 will have digital sums of 9. (A digital sum is the sum of the digits.) For example, 81 has a digital sum of 8 + 1 = 9, and 567 has a digital sum of 5 + 6 + 7 = 18, 1 + 8 = 9.

Figure It Out

Number Sense and Algebraic Thinking

Teachers' Notes

Overview of Levels 3–4: Book One

Title	Content	Page in students' book	Page in teachers' book
Speaking in Riddles	Solving problems with possibilities and constraints	1	14
The Great Factor Hunt	Using spreadsheets to find factors	2–3	16
The Factoring Factory	Using repeated factors and divisors	4–5	19
That Takes the Biscuit	Finding relationships between the areas of rectangles, parallelograms, and triangles	6–7	21
Stripping Fractions	Adding and subtracting fractions	8–9	24
What to Do?	Deciding what operations to perform in word problems	10–11	26
Pondering Percentages	Finding percentages of amounts	12–13	28
Close Ties	Exploring fractions	14–15	30
Fishy Fractions	Finding a fraction between two other fractions	16–19	32
Tile the Town, Tiny!	Estimating and calculating multiplication using multiples of 10	20–21	34
Wheeling and Dealing	Investigating digit patterns in multiples	22–24	36

Introduction to Number Sense and Algebraic Thinking

The *Number Sense and Algebraic Thinking* books in the Figure It Out series provide teachers with material to support them in developing these two key abilities with their students. The books are companion resources to Book 8 in the Numeracy Project series: *Teaching Number Sense and Algebraic Thinking*.

Number sense

Number sense involves the intelligent application of number knowledge and strategies to a broad range of contexts. Therefore, developing students' number sense is about helping them to gain an understanding of numbers and operations and of how to apply them flexibly and appropriately in a range of situations. Number sense skills include estimating, using mental strategies, recognising the reasonableness of answers, and using benchmarks. Students with good number sense can choose the best strategy for solving a problem and communicate their strategies and solutions to others.

The teaching of number sense has become increasingly important worldwide. This emphasis has been motivated by a number of factors. Firstly, the traditional approaches to teaching number have focused on preparing students to be reliable human calculators. This has frequently resulted in their having the ability to calculate answers without gaining any real understanding of the concepts behind the calculations.

Secondly, technologies – particularly calculators and computers – have changed the face of calculation. Now that machines in society can calculate everything from supermarket change to bank balances, the emphasis on calculation has changed. In order to make the most of these technologies, students need to develop efficient mental strategies, understand which operations to use, and have good estimation skills that help them to recognise the appropriateness of answers.

Thirdly, students are being educated in an environment that is rich in information. Students need to develop number skills that will help them make sense of this information. Interpreting information in a range of representations is critical to making effective decisions throughout one's life, from arranging mortgages to planning trips.

Algebraic thinking

Although some argue that algebra only begins when a set of symbols stands for an object or situation, there is a growing consensus that the ideas of algebra have a place at every level of the mathematics curriculum and that the foundations for symbolic algebra lie in students' understanding of arithmetic. Good understanding of arithmetic requires much more than the ability to get answers quickly and accurately, important as this is. Finding patterns in the process of arithmetic is as important as finding answers.

The term "algebraic thinking" refers to reasoning that involves making generalisations or finding patterns that apply to *all* examples of a given set of numbers and/or an arithmetic operation. For example, students might investigate adding, subtracting, and multiplying odd and even numbers. This activity would involve algebraic thinking at the point where students discover and describe patterns such as "If you add two odd numbers, the answer is always even." This pattern applies to *all* odd numbers, so it is a generalisation.

Students make these generalisations through the process of problem solving, which allows them to connect ideas and to apply number properties to other related problems. You can promote process-oriented learning by discussing the mental strategies that your students are using to solve problems. This discussion has two important functions: it gives you a window into your students' thinking, and it effectively changes the focus of problem solving from the outcome to the process.

Although the term "algebraic thinking" suggests that generalisations could be expressed using algebraic symbols, these Figure It Out *Number Sense and Algebraic Thinking* books (which are aimed at levels 2–3, 3, and 3–4) seldom use such symbols. Symbolic expression needs to be developed cautiously with students as a sequel to helping them recognise patterns and describe them in words. For example, students must first realise and be able to explain that moving objects from one set to another does not change the total number in the two sets before they can learn to write the generalisation $a + b = (a + n) + (b - n)$, where n is the number of objects that are moved. There is scope in the books to develop algebraic notation if you think your students are ready for it.

The Figure It Out *Number Sense and Algebraic Thinking* books

The learning experiences in these books attempt to capture the key principles of sense-making and generalisation. The contexts used vary from everyday situations to the imaginary and from problems that are exclusively number based to those that use geometry, measurement, and statistics as vehicles for number work. Teachers' notes are provided to help you to extend the ideas contained in the activities and to provide guidance to your students in developing their number sense and algebraic thinking.

There are six *Number Sense and Algebraic Thinking* books in this series:
Levels 2–3 (Book One)
Levels 2–3 (Book Two)
Level 3 (Book One)
Level 3 (Book Two)
Levels 3–4 (Book One)
Levels 3–4 (Book Two)

Page 1: Speaking in Riddles

Achievement Objectives

- write and solve problems which involve whole numbers and decimals and which require a choice of one or more of the four arithmetic operations (Number, level 3)
- devise and use problem-solving strategies to explore situations mathematically (Mathematical Process, problem solving, levels 3–4)
- use their own language, and mathematical language and diagrams, to explain mathematical ideas (Mathematical Processes, communicating mathematical ideas, levels 3–4)

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Number Framework Links

Use this activity to encourage transition from advanced additive strategies (stage 6) to advanced multiplicative strategies (stage 7).

Activity

This activity involves solving problems with possibilities and constraints. It's a useful activity for students with a good recall of basic multiplication and division facts who are learning to express whole numbers as the product of prime numbers. These students will need to be at the advanced additive stage at least for multiplication and division because the problem involves finding factors of 36.

This problem is a simplified version of a problem created by George Polya, a renowned mathematician, who was often called "the father of problem solving". It contains several clues about the ages of Albert's children that do not appear to be sufficient to answer the problem. But they are!

The important information the students need to identify is:

- The problem is asking them to find out each child's age.
- There are 3 children.
- The product of the 3 ages is 36.
- The sum of the ages is an odd number.
- 2 children are the same age (twins).
- The twins are the youngest.

The first clue is that the three children's ages multiply together to give a product of 36. Encourage the students to be systematic in finding all the possibilities for the ages. One way is to begin with 1 as a factor and list the possibilities: 1 x 1 x 36 (36 is rather old for a child), 1 x 2 x 18 (also an old child, but possible), 1 x 3 x 12, 1 x 4 x 9, and 1 x 6 x 6. (From the later clue about twins, which includes the fact that they are the youngest, we know that none of these are possibilities for the ages.) Note that once all the possibilities using a particular factor are exhausted, this narrows down the possibilities because that factor cannot be used again.

The students could then work on the possibilities with 2 as a factor, then 3 as a factor, and so on. This systematic search might lead to the following possibilities: 2 x 2 x 9, 2 x 3 x 6, and 3 x 3 x 4. Given the twins clue, only 2 x 2 x 9 and 3 x 3 x 4 might work, although the students may already have eliminated 3 x 3 x 4 because of the odd number clue. (3 + 3 + 4 adds up to 10, which is an even number. 2 + 2 + 9 adds up to 13.) So this leaves only 2 x 2 x 9.

Question **3** involves the idea of possibilities and constraints. In such problems, an idea creates a number of possibilities that are then restricted by some stipulation (constraint). When the students come to do question **3**, they may find it very difficult to know where to start and to know when they have finished. You could first go through the process with them, creating a whole-class problem. This will give your students the necessary scaffolding. It would also help them if you limit problems to those that involve just 3 (different) ages. Here is one example:

- Choose three "ages": 5, 11, 13
- Make a constraint (condition): The oldest person is 8 years older than the youngest person.
- Make a second constraint: All the ages are odd.
- Make a third constraint: The sum of the ages is 29.

Check that the problem is solvable (that is, 3 numbers can be found that meet all 3 constraints, and there is only 1 such set of numbers). This can be done using a table:

Constraint 1	Constraint 2	Constraint 3 (age of middle person)	Comment
1, 9	?	19 ✗	To give a total of 29, the age of the middle person would have to be more than the age of the oldest person!
2, 10	✗		
3, 11	?	15 ✗	
4, 12	✗		
5, 13	?	11 ✓	
6, 14	✗		To give a total of 29, the age of the middle person would have to be equal to or less than the age of the youngest person.
7, 15	?	7 ✗	
8, 16	✗		

Note that the missing third number means that pairs can only be eliminated or queried in the "constraint 2" column because it is not known if the third number is even or odd at this point.

Next, get the students to brainstorm different constraints that might be used for this same single set of numbers. There are many possibilities, including:

- The middle age is more than twice the youngest age.
- Each age is less than 15.
- The older two ages are only 2 years apart.
- The youngest and oldest ages add up to 18.
- All three ages are prime numbers.
- The two older ages add up to 24.
- The age of the oldest person is 3 less than the sum of the ages of the other two.

The students may wonder if 3 constraints are always enough for a 3-age problem to be solved. Depending on your group, you may like to let them explore this hypothesis themselves. In fact, 3 constraints should always be enough (as long as they contribute genuinely new information). Likewise, 2 constraints should be enough for a 2-age problem and 4 constraints enough for a 4-age problem.

Another example you could work through with the group is based on money. For example, *"Melanie has 65 cents in her hand, made up of coins"* opens up many possibilities, including six 10 cent coins and one 5 cent coin; one 50 cent, one 10 cent, and one 5 cent coin; thirteen 5 cent coins; and so on. Adding the constraint *"the amount is made up of five coins"* results in all but one of the possibilities being eliminated: two 20 cent coins, two 10 cent coins, and one 5 cent coin. The students may need help with organising possibilities. Again, a table or a systematic list is a good way to ensure that all of the possibilities are accounted for. For example, a partial table for Melanie's problem (above) might look like this:

50 cent	20 cent	10 cent	5 cent	Number of coins
1		1	1	3
1			3	4
	3		1	4
	2	2	1	5 ✓
	2	1	3	6
	2		5	7
	1	4	1	6
	1	3	3	7
	1	2	5	8
	1	1	7	9
	1		9	10

Once they have the idea, let your students create their own problems involving possibilities and constraints for each other to solve. This will provide them with an opportunity to think logically and systematically. If they are using 3 ages, suggest that they keep the ages fairly close to each other, at least until they are sure that they know what they are doing. This will limit the number of possibilities to be tested. You could suggest other suitable contexts, such as a group of animals with 2 legs and 4 legs in a pen or posting a letter costing $1 with possible stamps of 10 cents, 30 cents, 50 cents, and so on.

Resources to support this approach can be found in:
- www.nzmaths.co.nz/PS
- Ministry of Education (1999). *Teaching Problem Solving in Mathematics: Years 1–8.* Wellington: Learning Media.
- *Problem Solving*, Figure It Out, levels 2–3, 3, 3–4.

Pages 2–3: The Great Factor Hunt

Achievement Objectives
- recall the basic multiplication facts (Number, level 3)
- make up and use a rule to create a sequential pattern (Algebra, level 3)

Other mathematical ideas and processes
Students will also:
- explore number patterns that involve both whole numbers and decimals
- use a computer spreadsheet to explore factors.

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Number Framework Links
To attempt these activities, students will need to be able to use advanced multiplicative strategies (stage 7) or higher and have some experience in using computer spreadsheets, especially in using formulae or functions.

Activity One

The "hunt" in this activity has computer spreadsheets doing the calculation work in the search for factors. It allows the students to think about the number properties involved rather than attending to the demands of calculations.

Ensure that the students understand that factors of a number, for example, 16, are whole numbers that multiply to give that number. For 16, such numbers are 1 x 16, 2 x 8, and 4 x 4.

The students also need to understand the geometric properties of a spreadsheet: the cells are designated by co-ordinates, with a vertical reference (usually a letter) and a horizontal reference (usually a number).

The spreadsheet operation may function differently with some programs, so try the activity on the class computer yourself first. For information on the use of spreadsheets, see pages 26–27, *Answers and Teachers' Notes: Algebra*, Figure It Out, Level 3. Spreadsheet Challenge on pages 12–13 of *Algebra*, Figure it Out, Level 3 offers a supporting activity. Picking Patterns on pages 6–9 of *Number Sense and Algebraic Thinking*, Figure It Out, Levels 3–4, explains how to use the Fill Down function.

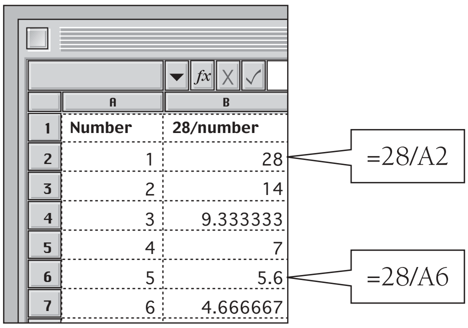
Students who don't have access to a computer could set up a spreadsheet on paper. The Fill Down function can be done manually using a calculator and a table. The students should be able to recognise the pattern: if a number is a factor, the quotient (answer) must be a whole number. (For example, if 8 is a factor of 16, the answer to 16 ÷ 8 will be a whole number. 16 ÷ 8 = 2, so 8 and 2 are factors of 16.)

Questions **1–3** reinforce this fact, that when a number is divided by one of its factors, the quotient is also a whole number. So, in general, to search for factors of a number, the students tell the computer to systematically work through dividing the number by the whole numbers from 1 to the number's square root. Where the division gives whole number answers, both the divisor (dividing number) and the quotient (answer) are factors. The example of Trapper Tino's search for factors of 10 on page **2** shows that for division by 1, 2, 5, and 10, the answer is a whole number. 1, 2, 5, and 10 are the whole-number factors of 10.

If the factor hunt is systematic and begins with 1 as a factor, the multiplication pairs will be listed below and above the square root of the number in question. For example, for 16, the pairs will be 1 and 16 and 2 and 8, so the list of factors will be 1, 2, 4, 8, and 16. ($\sqrt{16} = 4$, so 4 is also a factor.) Once the square root is reached, the search is complete. (After this, the factors simply repeat, but in the reverse order.) Be aware that for a number like 24, the square root isn't a whole number and therefore isn't a factor.

To search for the factors of 28 in question **3b**, the students will need to instruct the computer to divide 28 by the numbers 1 to 28. (The computer search could stop after dividing by 6 because the square root of 28 is less than 6. It's a good idea to allow the students to discover this for themselves.)

Setting up the spreadsheet as follows will solve the problem:



From the spreadsheet, the students can see that 1 x 28, 2 x 14, and 4 x 7 are the only pairs of factors of 28. So the list of factors of 28 is: 1, 2, 4, 7, 14, 28.

Note that for question **4**, Trapper Tino could divide 203 by all of the whole numbers from 1 to 203, but because the square root of 203 is 14.24781, he only needs to use the numbers 1 to 15. On the spreadsheet that goes with this question, Tino's search for the factors of 203 is shown to be incomplete because he has divided only by whole numbers up to 9. In fact, dividing 203 by the numbers 10 to 15 doesn't uncover any more factors. You might expect this because 3 in the ones digit of 203 means that dividing by any even number or a multiple of 5 could not work. That leaves 11 and 13 as the only possible whole numbers that might divide evenly into 203. They don't!

Note: To use the square root calculator on the spreadsheet, type in =SQRT(number) in any cell. For example, to calculate $\sqrt{49}$, type in =SQRT(49).

Activity Two

This activity extends the investigations of factors. The activity will reinforce knowledge of basic facts and is a good chance to practise with divisibility tests.

When looking for factors, the students should always work systematically from least to greatest and should remember that every factor has its pair (except that in the case of square numbers, the pair is the same number, for example, in 5 x 5 = 25). If they follow this approach, they will minimise the risk of overlooking a factor. The students will probably work better in pairs. Many students will need scaffolding to help them through these investigations. You could offer this as hints enclosed in envelopes, which the students can access if they wish. On the next page are some hints that might be useful for each investigation. The answers are in brackets. Don't make these answers available to the students.

Question **1** focuses on prime numbers. A prime number has only two factors, 1 and the number itself. 1 is not considered to be a prime number because it only has one factor – itself. Any number that has more than two factors is called a composite number. Note that the question states that apart from 2, all prime numbers are odd. This doesn't mean that all odd numbers apart from 1 are prime. For example, 11 is a prime number, but 33 isn't.

Hints for question **1**:

i. *Even numbers can all be divided by what number?* (2)
Why can't even numbers other than 2 be prime? (Because 2 will be one of their factors)

ii. *Why can't 99, 98, 96, 95, 94, and 93 be prime numbers?* (99, 96, and 93 can be divided by 3, 98 and 94 are even, and 95 has 5 as a factor.)

iii. *How many factors do prime numbers have?* (2 factors, 1 and themselves)
How many factors does 1 have? (1 factor, itself)

Question **2a** focuses on matching pairs of factors, while **2b** highlights the fact that square numbers are only listed once as a factor.

Hints for question **2**:

i. *The factors of a number come in pairs. For example, 1 x 24, 2 x 12, 3 x 8, and 4 x 6 give the 8 factors of 24. Does 25 have factor pairs like that?* (Yes, 1 x 25 and 5 x 5, but the list of factors is {1, 5, 25}.)

ii. *If you make squares with small square tiles, how many tiles does it take to make complete squares?* (1, 4, 9, 16, 25, 36, and so on. These are square numbers. For example, a 5 x 5 square takes 25 tiles.)

iii. *Imagine that Hunter Vaitoa Saluni used a spreadsheet to find the factors of 100. In the "divide by 10" row, the answer was 10. What does this say about 100?* (100 is a square number.) *Does 100 have an odd or even number of factors?* (It has an odd number of factors because the 10 from 10 x 10 is listed once only as a factor.)

Hints for question **3**:

i. *Could the number be a prime number? Why/why not?* (No. 1 + 83 = 84, and 83 is not between 50 and 70.)

ii. *What kinds of numbers between 50 and 70 will have too many factors to make a total of 84?* (Numbers like 54 and 64 that are multiples of 6 or 8 because they will have 2 and half of the number as one factor pair and 3 or 4 as factors as well. For example, the factors of 54 are 1, 2, 3, 6, 9, 18, 27, and 54.)

Hints for question **4**:

i. *The number has an odd number of factors. From question* **2***, what kind of a number is it?* (A square number. The square numbers less than 100 are 1, 4, 9, 16, 25, 36, 49, 64, and 81.)

ii. *How do you tell if a number is divisible by 3 (has 3 as a factor)?* (Its digits add to a multiple of 3, for example, for 69: 6 + 9 = 15, which is a multiple of 3.) See also "Wheeling and Dealing" in this student book, which further explores 3 and 9.

There is an infinite number of factor problems that the students could come up with for question **5**. To keep the maths manageable, you may find it helpful to limit the scope of the question to numbers less than 100. The students should look for a number that has several factors, and they should set a couple of constraints that are sufficient to identify it. They should then carefully check that there aren't any other numbers that meet those same constraints. If there are, they should tighten the constraints or add a further one before giving the problem to a classmate to solve.

Some students may find it helpful to use the Sieve of Eratosthenes, which is based on finding multiples of a given number. See the Numeracy Project material master 8-13 (available at www.nzmaths.co.nz/numeracy/materialmasters.htm).

Here are two possible problems that you could use as further examples:

- *Two prime numbers have a sum of 120 and a difference of 14. What are they?* (53 and 67)
- *A number less than 100 has 6 factors, of which 4 are divisible by 5. What is the sum of the factors?* (The number is 75, which has the factors {1, 3, 5, 15, 25, 75}. The factor sum is therefore 124.)

Extension

The students could find out more about the mathematical history of prime numbers. For example: *Goldbach, an eighteenth century German mathematician, declared that every even number except 2 is the sum of two prime numbers. Is this statement true?*

Numeracy Project materials (see www.nzmaths.co.nz/numeracy/project_material.htm)

Support teaching material on aspects such as squaring, square roots, and divisibility can be found in *Book 8: Teaching Number Sense and Algebraic Thinking*, pages 28–35.

Pages 4–5: The Factoring Factory

Achievement Objectives

- recall the basic multiplication facts (Number, level 3)
- state the general rule for a set of similar practical problems (Algebra, level 3)
- devise and follow a set of instructions to carry out a mathematical activity (Mathematical Processes, communicating mathematical ideas, levels 3–4)

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Number Framework Links

Use these activities to encourage transition from advanced additive strategies (stage 6) to advanced multiplicative strategies (stage 7) in the domain of multiplication and division.

Activities One and Two

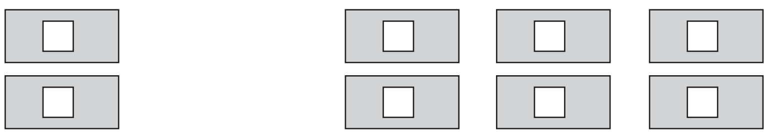
These activities require students to consider how one factor in a multiplication or the divisor in a division can be partitioned by factorising. At the advanced additive stage, students learn to partition one factor in a multiplication additively. For example, the student might solve 7 x 15 as 7 x 10 + 7 x 5. The 15 has been partitioned into 10 + 5. This is an example of the distributive property at work.

Advanced multiplicative students should recognise that they can also partition numbers by factors. For example, they could rewrite 8 x 17 as 17 x 8 and solve it using 2 x 2 x 2, which is a factorisation of 8. Using a doubling strategy, they could solve this easily as 17 x 2 x 2 x 2 (that is, 34, 68, 136). Similarly, they could solve 216 ÷ 12 as 216 ÷ 2 ÷ 2 ÷ 3 (that is, 108, 54, 18), where 2 x 2 x 3 is a factorisation of 12.

Knowing how to factorise is a key to understanding equivalent fractions and more complex work such as dividing fractions and decimals. For example, finding $\frac{1}{3}$ of $\frac{1}{5}$ (that is, $\frac{1}{3} \times \frac{1}{5} = \frac{1}{15}$) involves recognising that 3 x 5 is a factorisation of 15.

For a lesson on how to teach multiplicative partitioning by factors, refer to pages 38 to 40, *Book 6: Teaching Multiplication and Division* in the Numeracy Project series.

Students who require support with this concept will find strip diagrams useful for imaging. For example, to find the effect of multiplying a number by 2 then 3, you could draw:



2 x $\square$ 3 x 2 x $\square$ is the same as 6 x $\square$.

A similar diagram could be used to show the effect of dividing a number by 2 and then dividing the result by 3:

?

? ÷ 2	? ÷ 2

? ÷ 2 ÷ 3	? ÷ 2 ÷ 3	? ÷ 2 ÷ 3	? ÷ 2 ÷ 3	? ÷ 2 ÷ 3	? ÷ 2 ÷ 3

$\square$ ÷ 2 ÷ 3 is the same as $\square$ ÷ 6 (because it creates $\frac{1}{6}$ of the original number).

Question **4** in **Activity Two** directs the students' attention towards prime number factors. The fundamental theorem of arithmetic says that each whole number has a single, unique factorisation made of prime numbers. For example, you can write 30 as the product of prime numbers in only one way, 2 x 3 x 5, and for 14, the prime factorisation is 2 x 7.

In the context of Shaun's replicating machine, only the prime number settings are necessary. The x 4 and x 6 settings are not needed because the same effect can be produced using prime number factors: x 2 followed by x 2 results in x 4 (just as ÷ 2 followed by ÷ 2 results in ÷ 4), and x 2 followed by x 3 results in x 6 (just as ÷ 2 followed by ÷ 3 results in ÷ 6). Also, x 3 followed by x 3 results in x 9, and ÷ 3 followed by ÷ 3 results in ÷ 9. If Shaun added 7, 11, and 13 to the dials, he could then create a large number of replications using prime number factors. For example, to produce 110 times anything, he could set the machine at x 2, then x 5, then x 11.

Extension

The students could investigate problems in which prime number factorisation allows all of the possible factors of a number to be found. For example, *The Juice Company wants to design a 330 millilitre carton that fits nicely into people's hands. They want the container to have edges that measure to an exact number of centimetres. What cuboid-shaped cartons could they make? Which one would fit a hand best?*

The prime number factorisation of 330 is 2 x 3 x 5 x 11. Regrouping these factors gives 6 x 5 x 11, 2 x 3 x 55, 2 x 15 x 11, 10 x 3 x 11, 22 x 3 x 5, and 33 x 2 x 5. These factors give the possible dimensions of the carton in centimetres. (1 could also be considered as a factor, but that would make a very thin carton.)

The students could also investigate the significance of prime number factors in finding the simplest form of equivalent fractions. For example, consider the problem *"What is the simplest fraction equivalent to $\frac{60}{84}$?"* One approach is to divide both the numerator (top number) and denominator (bottom number) by the same or progressively larger prime numbers. So $\frac{60}{84} = \frac{30}{42}$ (dividing by 2), $\frac{30}{42} = \frac{15}{21}$ (dividing by 2), and $\frac{15}{21} = \frac{5}{7}$ (dividing by 3). When this process gives a numerator or a denominator that is a prime number (both are in this case), the fraction can't be simplified further.

Numeracy Project materials (see www.nzmaths.co.nz/numeracy/project_material.htm)

- *Book 7: Teaching Fractions, Decimals, and Percentages*
  Trains, page 19.

Pages 6–7: That Takes the Biscuit

Achievement Objectives

- state the general rule for a set of similar practical problems (Algebra, level 3)
- calculate perimeters of circles, rectangles, and triangles, areas of rectangles, and volumes of cuboids from measurements of length (Measurement, level 4)
- use their own language, and mathematical language and diagrams, to explain mathematical ideas (Mathematical Processes, communicating mathematical ideas, levels 3–4)
- classify objects, numbers, and ideas (Mathematical Processes, developing logic and reasoning, levels 3–4)

AC
EA
AA
AM
AP

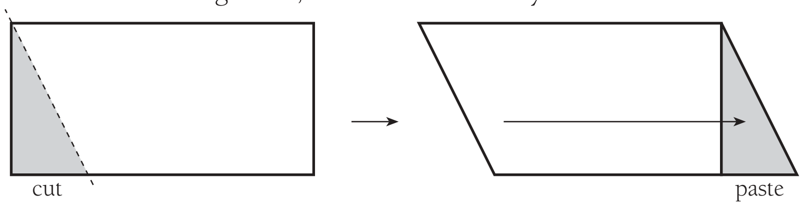
Number Framework Links

This activity is suitable for students using advanced counting strategies (stage 4) or higher. Students using advanced additive strategies (stage 6) or higher will benefit most from questions **6** and **7** because these questions require multiplicative strategies.

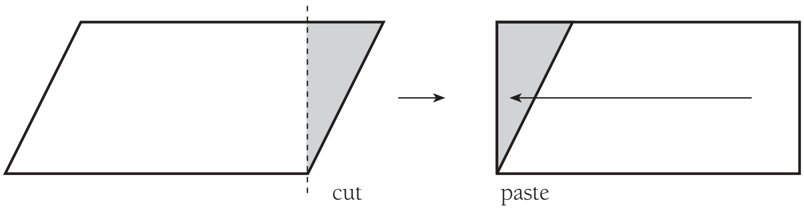
Activity

This activity invites students to generalise how to find the areas of parallelograms and triangles from the area of the surrounding rectangle. Multiplicative strategies are useful for this activity but are not essential. The reasoning required here is mostly spatial, although questions **6** and **7** invite some use of numbers.

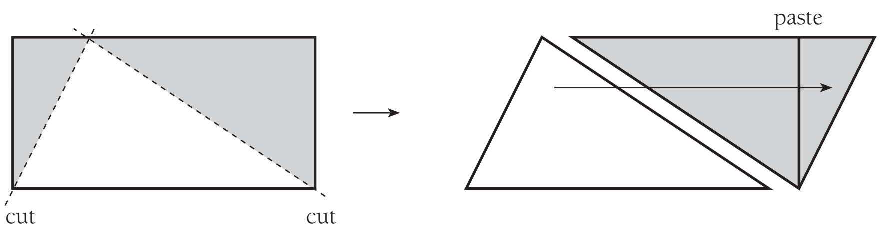
By cutting and pasting, it is possible to change any rectangle to a parallelogram of the same area and vice versa. In general, it is done in this way:



or



Similarly, it is possible by cutting and pasting to show that a triangle has half the area of its surrounding rectangle, and likewise, that a rectangle has twice the area of any triangle it surrounds, as long as the triangle and rectangle share 2 vertices (have the same base). Diagrammatically, the cutting and pasting looks like this:



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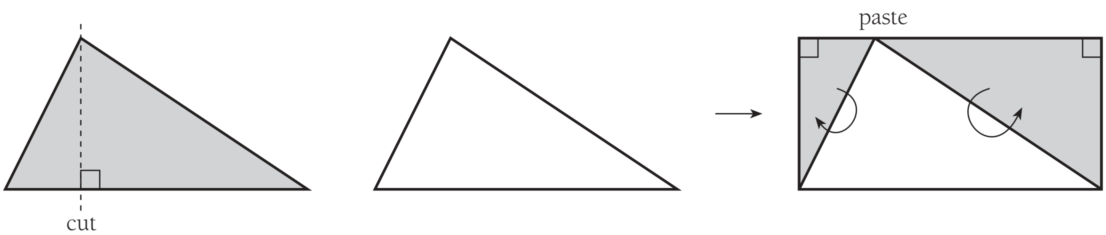
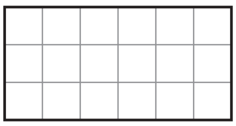


Questions **1**, **2**, and **3** require the students to generalise these properties through spatial visualisation. The students' ability to directly compare the areas of rectangles and parallelograms (see question **2**) will indicate their understanding of the attribute of area. In other words, do they recognise that shapes can be compared by area? For teaching ideas and student progressions in area, you could refer to pages 3–5 and 11–13 of *Book 9: Teaching Number through Measurement, Geometry, Algebra, and Statistics* in the Numeracy Project series.

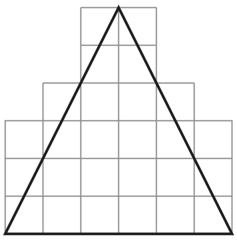
Question **2a** deals with two rectangles rather than one rectangle and one parallelogram, and they don't have the same height or base, so the principle stated above does not apply. The shapes do, however, have the same area, and this can be shown by cutting and rearranging. In general, different-shaped rectangles don't have the same area. Challenge your students to discover what there is about the relationship between these two rectangles that means that they do have the same area. (Poochpower is $\frac{2}{3}$ the height of Barking Mad and $1\frac{1}{2}$ times its length. Mathematically, this relationship can be represented like this: $\frac{2}{3} \times \frac{3}{2} = 1$.)

Note that most of this activity is about direct comparison of area rather than the use of a unit to measure it. Don't assume that your students have a unit concept for area simply because they can directly compare. Frequently, students do not understand that squares are used as a unit because they tessellate. They will often want to use non-tessellating shapes like beans or counters when the shape to be measured has curved sides. They may not be aware that in measuring area, the square units are located in rows and columns so the amount can be calculated easily with multiplication.

This situation becomes more complex when the shape is made up of whole and part square units, as with triangles and parallelograms. When the students answer questions **4–7**, encourage them to apply the area principles they have established for rectangles and parallelograms. This may help them to establish how square units can be partitioned and recombined. For example, in question **5b**, extend the activity to include unit measurement by drawing diagrams such as the following:

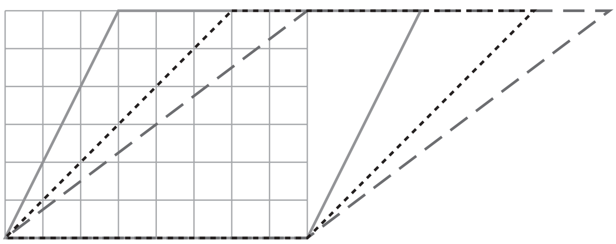


This rectangle has an area of 18 squares.



What is the area of this triangle?
How do you know?

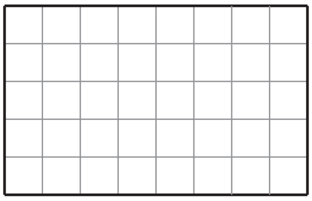
Look for the students to establish spatially that parts of squares can be recombined to form whole units. This will help them to understand questions **6** and **7** better. In question **6**, look for the students to begin with "parent" rectangles of 48 and 24 squares. From these rectangles, they can make an infinite number of correct parallelograms and triangles. For example, consider question **6a**:



An 8 x 6 rectangle has an area of 48 squares, so these parallelograms will also have that area.

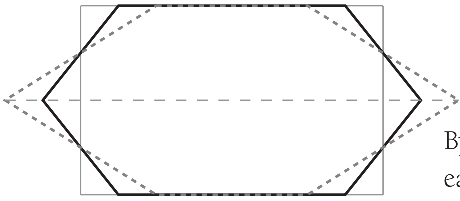
Similarly for question **7**, look for your students to reason spatially about how to create hexagons and octagons with the same area as a rectangle with an area of 40 squares. This rectangle could be 5 x 8, 4 x 10, 2 x 20, 1 x 40, or $2\frac{1}{2}$ x 16. The question doesn't specify a regular hexagon or octagon, so the side lengths don't have to be all the same. In the case of the hexagon, the students will need to reason how they can transform a rectangle of area 40 into a hexagon with the same area.

Step 1:



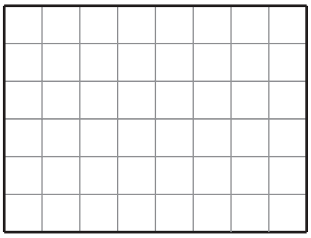
An 8 x 5 rectangle has an area of 40 squares.

Step 2:

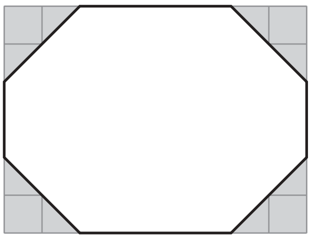


By using symmetry, an infinite number of hexagons, each with an area of 40 squares, can be created.

With the octagon, the students will need to consider how a rectangle with an area greater than 40 squares can be reduced to form an octagon with an area of 40 squares. This diagram shows one solution:

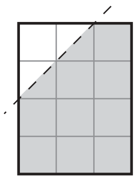


Starting with a 6 x 8 rectangle that has an area of 48 squares …

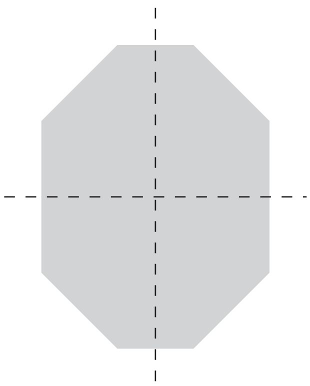


Cutting off 4 triangles with areas of 2 squares each creates an octagon with an area of 40 squares.

Another strategy could be to make $\frac{1}{4}$ of an octagon with an area of 10 squares and then use symmetry to make a complete octagon with an area of 40 squares. The diagram below shows one such solution:



Form a 4 x 3 rectangle and cut off a triangle with an area of 2 squares.

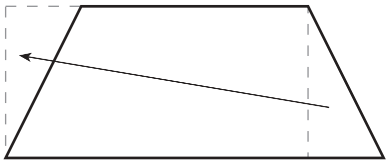


Reflect the quarter octagon horizontally and vertically to complete the shape.

$\frac{1}{4}$ of an octagon with an area of 40 squares will have an area of 10 squares.

Extension

The students could investigate how to find the areas of other shapes. The trapezium is a good example. A trapezium is a quadrilateral with only 1 pair of parallel sides. Challenge the students to find the connection between the area of a trapezium and the area of rectangles of a related size.



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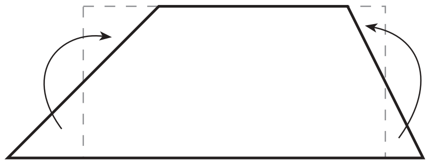
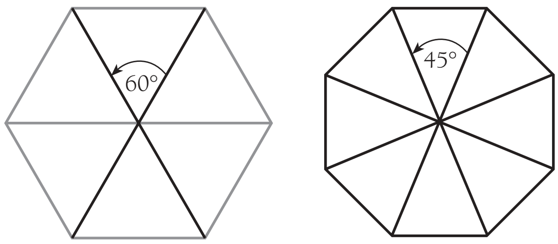


The students could also use their knowledge of triangles and angles to create regular hexagons and octagons:



Pages 8–9: Stripping Fractions

Achievement Objectives

- find fractions equivalent to one given (Number, level 4)
- use their own language, and mathematical language and diagrams, to explain mathematical ideas (Mathematical Processes, communicating mathematical ideas, levels 3–4)

AC
EA
AA
AM
AP

Number Framework Links

Use this activity to:

- help students to use advanced additive strategies (stage 6) or advanced multiplicative strategies (stage 7)
- encourage transition to advanced proportional strategies (stage 8). The activity is appropriate for these levels because it requires an understanding of equivalent fractions.

Activity

Questions **1–4** in this activity involve adding and subtracting fractions, including improper fractions (that is, fractions in which the numerator is equal to or greater than the denominator, for example, $\frac{7}{5}$). The students need to be able to rename improper fractions as mixed numerals (for example, $\frac{3}{2}$ as $1\frac{1}{2}$). The key to these operations is understanding equivalent fractions.

If addition and subtraction of fractions is taught before multiplicative understanding is established, students tend to develop misconceptions, often naively transferring rules for whole numbers to fractions. Common errors include:

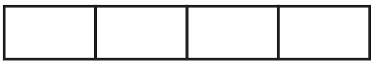
- adding denominators as well as numerators, for example, $\frac{3}{4} + \frac{3}{4}$ as $\frac{3+3}{4+4} = \frac{6}{8}$ instead of $\frac{6}{4}$
- ordering fractions by the "gap" between the denominator and numerator, for example, thinking that $\frac{3}{5}$ is greater than $\frac{3}{4}$ because 5 – 3 = 2 (for $\frac{3}{5}$) is greater than 4 – 3 = 1 (for $\frac{3}{4}$).

Equivalent fractions can be taught by building up or splitting. Building up involves using materials to make a given fraction and looking for patterns in those fractions that have equal parts. For example, building up $\frac{3}{4}$ might establish $\frac{6}{8}$, $\frac{9}{12}$, and $\frac{12}{16}$ as equivalent fractions. You could invite the students to look for patterns. This should lead to the observation that there is a multiplicative connection between such fractions, for example:

$\frac{3}{4} = \frac{9}{12}$. The denominator and numerator are both multiplied by 3.

Finding equivalent fractions by splitting involves taking a fraction, splitting it into a given number of equal parts, and finding the size of each part. For example, 1 whole split into thirds:

Then $\frac{1}{3}$ cut into quarters gives $\frac{4}{12}$.



This requires the students to use whole-to-part and part-to-whole reasoning. Mapping back to 1 (whole) is difficult for students who don't have multiplicative understanding.

This activity uses fraction strips for adding and subtracting fractions. Fraction strips can be purchased commercially, or the students can make their own paper versions using the Numeracy Project material master 7-7 (available at www.nzmaths.co.nz/numeracy/materialmasters.htm) or the copymaster at the end of these notes.

Before the students attempt the activity, establish the principle that fractions are made up of duplications (repeats) of a unit fraction. (A unit fraction is a fraction that has a numerator of 1.) For example, $\frac{4}{5} = \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5}$, that is, 4 lots of $\frac{1}{5}$. Once they understand this principle, the students can understand that when fractions with the same denominators are joined, the answer is always in terms of duplications of the unit fraction. For example, $\frac{3}{8} + \frac{6}{8} = \frac{9}{8}$ is just 3 + 6 in lots of $\frac{1}{8}$ and is structurally identical to 300 + 600 = 900. The same principle holds true for subtraction, for example, $\frac{4}{5} - \frac{2}{5}$ is 4 – 2 in lots of $\frac{1}{5}$, giving the answer $\frac{2}{5}$.

Matters become more complicated where the fractions have different denominators because the unit fractions that make up the fractions are different sizes (see questions **5** and **6**). Sometimes the unit fractions are closely related, like halves and quarters, thirds and ninths (where one is a multiple of the other). This makes the task of converting both fractions to equivalent forms with the same denominator much easier. For example, for $\frac{7}{10} + \frac{3}{5}$, the $\frac{3}{5}$ can be converted to $\frac{6}{10}$, so the problem becomes $\frac{7}{10} + \frac{6}{10} = \frac{13}{10}$, which is $1\frac{3}{10}$.

Where the fractions being added or subtracted have unrelated denominators (that is, one is not a multiple of the other), both fractions need to be converted to equivalent fractions with a common denominator. The easiest denominator to use in converting the fractions is the lowest common multiple of the fractions' denominators. For example, with $\frac{3}{4} + \frac{2}{3}$, the multiples of 4 and 3 are 12, 24, 36, and so on, so both fractions could be renamed as so many twelfths, twenty-fourths, or thirty-sixths. 12 is the lowest common multiple. $\frac{3}{4} = \frac{9}{12}$, and $\frac{2}{3} = \frac{8}{12}$. So $\frac{3}{4} + \frac{2}{3}$ has the same answer as $\frac{9}{12} + \frac{8}{12}$, which is 9 + 8 = 17 twelfths ($\frac{17}{12}$ or $1\frac{5}{12}$). This also holds for subtraction, for example,

$$\begin{aligned} \tfrac{3}{4} - \tfrac{2}{3} &= \tfrac{9}{12} - \tfrac{8}{12} \\ &= \tfrac{1}{12}. \end{aligned}$$

Extension

The students could investigate why decimals were invented. (This was to make operations with fractions, particularly addition and subtraction, easier.) They could compare the difficulty of adding and subtracting fractions that have unrelated denominators with the difficulty of converting the fractions to decimals and adding or subtracting them.

Numeracy Project materials (see www.nzmaths.co.nz/numeracy/project_material.htm)

For questions **1–4**:

- *Book 7: Teaching Fractions, Decimals, and Percentages*
  Trains, page 19
- *Book 8: Teaching Number Sense and Algebraic Thinking*
  Fractions Greater than 1, page 17

For questions **5–6**:

- *Book 7: Teaching Fractions, Decimals, and Percentages*
  Trains, page 19 (which uses multiplication and division to compare the size of fractions with whole numbers, especially fractions greater than 1)
- *Book 8: Teaching Number Sense and Algebraic Thinking*
  Fraction Number Lines, page 18 (which shows mixed fractions and improper fractions on a double number line)
  Adding and Subtracting Fractions, page 41

Other resources

Figure It Out activities such as Fraction Frenzy, *Number: Book 3*, Figure It Out, level 3, pages 22–23.

Pages 10–11: What to Do?

Achievement Objectives

- write and solve problems which involve whole numbers and decimals and which require a choice of one or more of the four arithmetic operations (Number, level 3)
- solve problems of the type $\square + 15 = 39$ (Algebra, level 3)
- interpret information and results in context (Mathematical Processes, developing logic and reasoning, levels 3–4)

AC
EA
AA
AM
AP

Number Framework Links

Use this activity to encourage transition from advanced additive strategies (stage 6) to advanced multiplicative strategies (stage 7).

Activity

This activity encourages students to represent word problems as box diagrams and equations. Some educators refer to this as "mathematising". Research indicates that students often find mathematising difficult and need considerable support to analyse the arithmetic structures involved.

Be aware that different word problems that may be represented with the same equation can vary considerably in difficulty. The easiest problems for students to represent are usually those that describe actions. *"Sarah has 43 marbles and loses 18 of them. How many marbles has she got now?"* suggests an action of taking away. Students usually associate taking away with subtraction and can mathematise this as 43 – 18 = number.

However, some story problems have wording that don't suggest an action. For example, *"Kellie has 76 stamps, and Tyler has 49 stamps. How many more than Tyler does Kellie have?"* does not describe an action. It describes a difference, and the students must understand what action is needed to find that difference.

Other variables that affect the difficulty of word problems are the position of the unknown and the number of steps involved in the calculation. Problems that have the unknown at the start are hardest to solve. For example, *"Leighton has a lot of matchbox cars. After opening his birthday presents, he has 48 cars. That's 3 times as many cars as he had before. How many cars did he have before his birthday?"* Start-unknown problems are tricky because the student has to reverse the operation to solve them. In this example, the student needs to reverse "3 times as many", that is, reverse $\square \times 3 = 48$ to become $48 \div 3 = \square$.

Multiple-step problems are always harder to solve than single-step problems of the same kind because there are more pieces of information to organise and store during the calculation. The students need to be taught recording strategies that will help them to organise and store the important information in the problems they are trying to solve.

Box diagrams are powerful representations for a range of problems. The students need to be as accurate as possible with the ratios they allocate to each part of their diagram. Using square grid paper can help with this. Once the students have drawn a box diagram for any of the problems in this activity, get them to check that it matches the information. For example, suppose the students have presented the following diagram for question **5**:

52			
Alana	Brenda	Caleb	Donald

Alana, Brenda, Caleb, and Donald are the same age. Are the boxes for their ages the same size?
The 52 candles show their combined age on their next birthday. Do you have a single box labelled 52 that matches the total length of all four children's ages?
How will you work out what each box showing the children's ages is worth?
How old are the children now?

From the box diagram, the students will find it easier to solve the problem by operating on numbers, in this case, 52 ÷ 4. The catch in this question is "How old are they now?" 52 ÷ 4 gives 13 as their age on their next birthday, so they are 12 now.

The problems in questions **3–8** become increasingly complex because they involve inactive language and more calculation steps. For example, consider a possible box diagram sequence for question **8**:

There are 38 people on the train.

38

The train has 3 carriages.

38		
? Front	? Middle	? Last

The last carriage has 8 more people than the front carriage.

38			
?	?	8	?

The middle carriage has 5 fewer people than the last carriage. That means it must have 3 more people than the front carriage.

38				
?	3	?	8	?

Representing the box diagram as a calculation requires algebraic reasoning. The diagram shows that there are 3 unknowns of the same size, labelled "?", that represent the number of people in the front carriage. So the equation is $(3 \times \square) + 11 = 38$. The students might solve this by trial and improvement. Putting 10 as the unknown gives $3 \times 10 + 11 = 41$, which is too large. By adjusting the unknown value, the students will find the correct number, 9. Some students may recognise that the problem can be solved using inverse operations: $3 \times \square + 11 = 38$ so $3 \times \square = 27$ so $\square = 9$ in the front carriage. This shows a structural understanding of the operations involved. Another way of solving this is shown in the Answers.

Extension
Extend the use of box diagrams to problems involving fractions, ratios, and proportions. For example, *"Hine has lived one-third of her life in Kaikohe, one-quarter of it in Tīrau, one-sixth of it in Maketū, and the other 6 years in Pātea. How old is Hine?"*

An initial box diagram of the problem might look like this:

Hine's age			
$\frac{1}{3}$ Kaikohe	$\frac{1}{4}$ Tīrau	$\frac{1}{6}$ Maketū	6 years Pātea

Then:

<table>
<tr><td colspan="12">Hine's age</td></tr>
<tr><td>$\frac{1}{12}$</td><td>$\frac{1}{12}$</td><td>$\frac{1}{12}$</td><td>$\frac{1}{12}$</td><td>$\frac{1}{12}$</td><td>$\frac{1}{12}$</td><td>$\frac{1}{12}$</td><td>$\frac{1}{12}$</td><td>$\frac{1}{12}$</td><td>$\frac{1}{12}$</td><td>$\frac{1}{12}$</td><td>$\frac{1}{12}$</td></tr>
<tr><td colspan="4">$\frac{1}{3}$: Kaikohe</td><td colspan="3">$\frac{1}{4}$: Tīrau</td><td colspan="2">$\frac{1}{6}$: Maketū</td><td colspan="3">6 years : Pātea</td></tr>
</table>

From this, it can be seen that 6 years = $\frac{3}{12}$. $\frac{1}{12}$ is therefore 6 ÷ 3 = 2 years, so Hine must be 12 x 2 = 24 years old.

You can also work out Hine's age using equations:

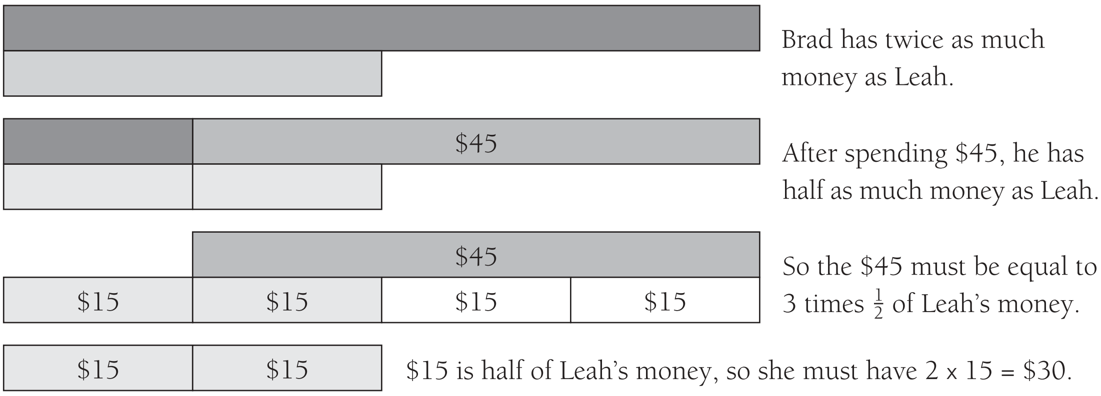
$\frac{1}{3} + \frac{1}{4} + \frac{1}{6} = \frac{4}{12} + \frac{3}{12} + \frac{2}{12}$

$= \frac{9}{12}$.

So 6 years is $\frac{12}{12} - \frac{9}{12} = \frac{3}{12}$.

6 years = $\frac{3}{12}$, so $\frac{1}{12}$ is 6 ÷ 3 = 2 years, so Hine must be 12 x 2 = 24 years old.

More difficult problems often involve change. For example, *"Brad has twice as much money as Leah has. He spends \$45. Now he has half as much money as Leah has. How much money does Leah have?"* Box diagrams are a good way to show this problem:



Pages 12–13: Pondering Percentages

Achievement Objectives

- express quantities as fractions or percentages of a whole (Number, level 4)
- find a given fraction or percentage of a quantity (Number, level 4)
- use their own language, and mathematical language and diagrams, to explain mathematical ideas (Mathematical Processes, communicating mathematical ideas, levels 3–4)

AC
EA
AA
AM
AP

Number Framework Links

Students will need to be able to use advanced multiplicative strategies (stage 7) or higher to work on this activity independently.

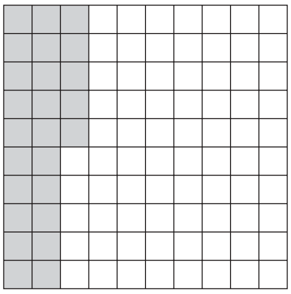
Activity

This activity gets students to represent fractions as percentages and to find percentages of money amounts using double number lines and 10 by 10 grids.

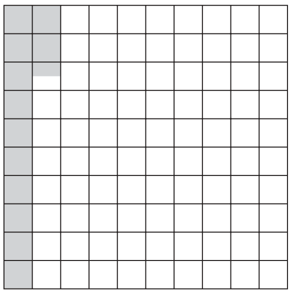
Decimals and percentages are special cases of equivalent fractions. For example, 75 percent is equivalent to $\frac{75}{100}$, which is equivalent to $\frac{3}{4}$ and many other fractions, such as $\frac{6}{8}$, $\frac{15}{20}$, and $\frac{30}{40}$. Percentages can be used either as operators (scalars), to enlarge or reduce an amount by a given proportion, or as proportions in their own right. Just like fractions, they are used as both operators and numbers. For example, "20 percent of 35" treats 20 percent as an operator, and "$\frac{6}{24}$ is equivalent to $\frac{25}{100}$ or 25 percent" treats 25 percent as a number.

The first part of the activity treats percentages as numbers. It uses the hundredths grid to convert common fractions like $\frac{1}{5}$, $\frac{1}{4}$, and $\frac{1}{10}$ to percentages. The students can draw 10 by 10 grids in their exercise books and shade the appropriate fraction of the grid. Alternatively, you could give them cut-out squares to fold into the fractional parts.

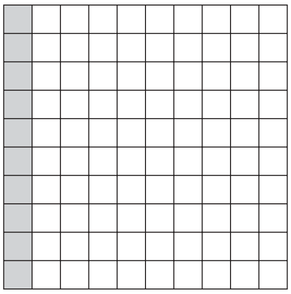
Encourage the students to look for connections between the fractions and the corresponding percentages. Consider these connections:



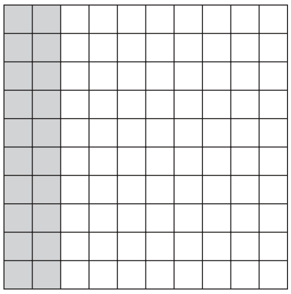
$\frac{1}{4}$ is 25 percent, so



$\frac{1}{8}$ is 12.5 percent (half as much).



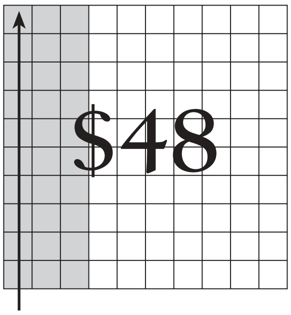
$\frac{1}{10}$ is 10 percent, so



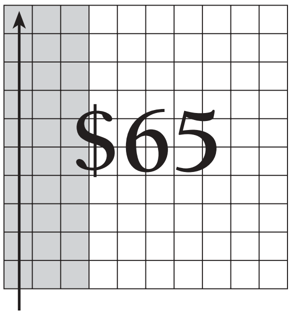
$\frac{1}{5}$ is 20 percent (twice as much).

From their experiences with percentages in contexts such as test marks and sports statistics, many students believe that percentages cannot be larger than 100, since 100 percent represents 1 or the whole. However, there are many real-life contexts, like lambing and calving rates and investment growth, in which percentages greater than 100 do occur. (If a farmer gets twice as many lambs as he has ewes, the lambing percentage is 200 percent.) Encourage the students to generalise this: 1 = 100 percent, so 2 = 200 percent, so 4 = 400 percent, and so on.

Question **2** involves using percentages as operators. The students' book uses a hundredths grid to show this. If, for example, 30 percent of a grid is shaded, the grid can be used to find 30 percent of any amount:



10% of 48 is 4.80 so 30% is 3 x 4.80



10% of 65 is 6.50 so 30% is 3 x 6.50

Note that with the hundredths grid, the value of each small square, or hundredth, of the amount can be calculated if necessary and is equivalent to 1 percent. For example, 1 percent of $65 is 0.65 (65 cents) because 65 ÷ 100 = 0.65. So 32 percent of $65 could be calculated by 0.65 x 32 or by adding 0.65 + 0.65 to 30 percent of $65.

A double number line can also be used to find a percentage of any amount. For example, 30 percent of $48 can be shown as:

Double number lines allow the students to solve difficult percentage problems by connecting and co-ordinating more manageable calculations. The double number line reflects the multiplicative relationship between the fraction and the percentage. For example, consider the problem "35 percent of $56". The order of calculation might be: 10 percent of 56 is 5.60, so 30 percent of 56 is 3 x 5.60 = 16.80, and 5 percent of 56 is $\frac{1}{2}$ of 5.60 = 2.80.
35 percent is 30 percent + 5 percent = $19.60.

For more ideas on how to develop double number lines to solve percentage problems, refer to pages 31–34 of *Book 7: Teaching Fractions, Decimals, and Percentages* in the Numeracy Project series.

Extension
To challenge the students further, pose problems where the conversions to percentages are not tidy and the students need to approximate the answer. You can do this by making the percentage and the money amount untidy, for example, *"find 67 percent of $47.95"*. In real-life situations, it is often enough to estimate the answers to such problems. 67 percent is about two-thirds, $47.95 is about $48.00. $\frac{2}{3}$ of 48 is 32. So $32 is a close estimate.

Encourage the students to use rounding and their knowledge of percentage to fraction conversions to help them estimate the answers to percentage problems.

Numeracy Project materials (see www.nzmaths.co.nz/numeracy/project_material.htm)
Using double number lines or percentage strips:
- *Book 4: Teaching Number Knowledge*
  Equivalent Fractions, Decimals, and Percentages, page 21
- *Book 7: Teaching Fractions, Decimals, and Percentages*
  Fractional Blocks, page 15
  Deci-mats, page 25
  Hot Shots, page 31

Pages 14–15: Close Ties

Achievement Objectives
- explain the meaning of the digits in decimal numbers with up to 3 decimal places (Number, level 3)
- make sensible estimates and check the reasonableness of answers (Number, levels 3–4)
- find fractions equivalent to one given (Number, level 4)

AC
EA
AA
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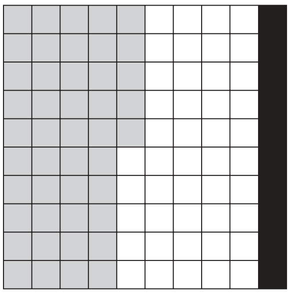
Number Framework Links
This activity is suitable for students using advanced multiplicative strategies (stage 7). Students will need to be at this stage or higher in the domains of multiplication and division if they are to work independently.

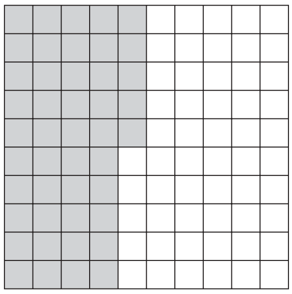
Activity

This activity is designed to encourage students to acquire number sense about the relative size of fractions. 0, $\frac{1}{2}$, and 1 are used as benchmarks, and they are very useful for estimation problems with fractions where only approximate answers are required. To have a strong understanding about the size of fractions, the students need to consider the multiplicative relationship between the numerator (top number) and denominator (bottom number) of the fraction. Some students assume that fractions can only be numbers that are less than 1. You may need to introduce or revise the idea of "improper fractions", for example, $\frac{13}{4} = 3\frac{1}{4}$, where the fraction describes 1 single whole.

Question **1a** asks the students to consider whether $\frac{45}{98}$ is closer to 0, $\frac{1}{2}$, or 1. The denominator suggests that 1 has been cut into ninety-eighths, that is, 98 equal parts. Students who are having difficulty could work through the problem using a hundredths grid.

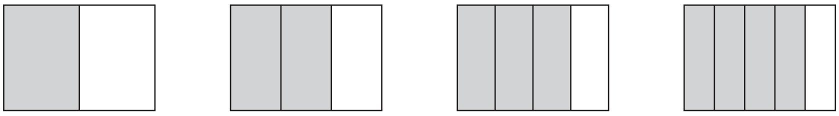


45 is half of 90, so 45 will be a bit less than half of 98.



45 out of 100 is close to 50 percent or 0.5 or $\frac{1}{2}$, so 45 out of 98 must also be close to $\frac{1}{2}$.

Kimiora's speech bubble reflects an additive comparison of the denominator and numerator. This "gap" thinking is common among additive thinkers and can be challenged by counter example. Start with fractions that have small denominators that the students may already know. Test the theory that a large gap between the numerator and denominator means that the fraction is small and therefore, a small gap means that the fraction is large. Consider $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{4}{5}$. They all have the same gap of 1 but get progressively larger and closer to 1.



Try other fraction combinations that have as large a gap between the numerator and denominator as $\frac{45}{98}$. For example, $\frac{147}{200}$ and then $\frac{447}{500}$. The students could make wafer bars with 200 and 500 little squares in them and shade in 147 and 447 of them. The Table function in a word processing program is good for this because you can create a 1-cell table that is full page size and then use the Split Cells function to divide it into as many parts as you want. You can then do shading in blocks. This way, the size of 1 stays constant. The students will soon find that about $\frac{3}{4}$ of the 200 square wafer and about $\frac{7}{8}$ of the 500 square wafer is shaded. After this, the students should accept that nearly half of the squares in the hundredths grid for the $\frac{45}{98}$ problem will be shaded.

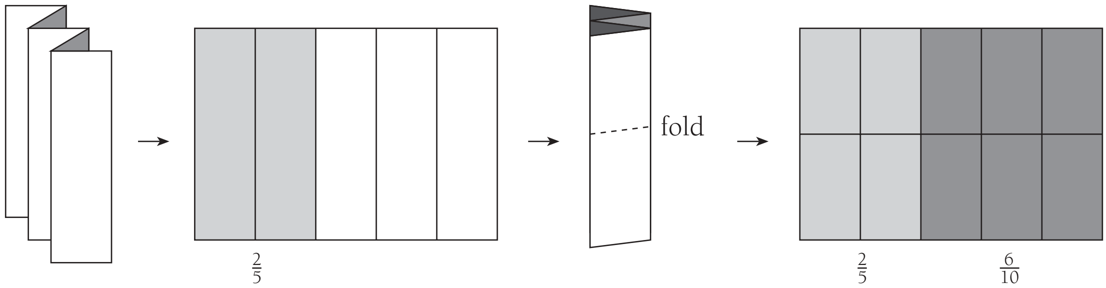
In question **1b**, the size of the numerator and denominator are deliberately pushed up to encourage the use of number properties. The students won't have ready access to physical models to decide whether $\frac{999}{10\,000}$ is closer to 0, $\frac{1}{2}$, or 1. Questions that might be used to scaffold student thinking could be:
What tidy number is 999 very close to? (1 000)
About how many times will 999 go into 10 000? (10)
Do you know any other fractions where the denominator is 10 times as big as the numerator? How big are those fractions? ($\frac{222}{2\,220}$, $\frac{333}{3\,330}$, and so on. Each fraction is $\frac{1}{10}$.)

So for question **1b**, note that only one of the speech bubbles (Kimiora's) shows correct reasoning (although Ruth makes a good start). A student who agrees with either of the first two speech bubbles is probably still thinking that the size of a fraction is determined by the size of either the numerator or the denominator.

It's not really a good idea to let the students use a calculator to confirm the proximity of a fraction to 0, $\frac{1}{2}$, or 1. Using a calculator not only undermines the development of sound number sense, it also, at the same time, makes assumptions about the relationship between the division operation and fractions and decimals.

In questions **4** and **5**, the students need to make deductions about the sum of two fractions based on their understanding of fraction size. Students who have a good understanding of equivalent fractions will know that $\frac{8}{16}$ equals $\frac{1}{2}$, so $\frac{7}{16}$ is less than $\frac{1}{2}$. This means that, in question **4**, $\frac{7}{16} + \frac{1}{2}$ must be less than 1.

Question **5** is more difficult because it involves the size of two fractions, one more than $\frac{1}{2}$ (that is, $\frac{6}{10}$) and one less than $\frac{1}{2}$ (that is, $\frac{2}{5}$). Again, the students need to use equivalent fractions. A piece of paper can be used as a model. The students can fold the paper into 5 equal parts and then shade in $\frac{2}{5}$ (2 fold sections). If they then fold the paper once, lengthwise, which gives tenths, and shade $\frac{6}{10}$, they will see that the whole piece of paper is shaded, showing that $\frac{2}{5} + \frac{6}{10} = 1$.



Extension
Extend the students by posing more difficult problems that require them to apply benchmarks such as 0, $\frac{1}{2}$, 1, $1\frac{1}{2}$, and so on. Good examples are:
$\frac{7}{12} + \frac{6}{13}$: Is the answer smaller than, equal to, or larger than 1?
$\frac{9}{10} - \frac{3}{8}$: Is the answer smaller than, equal to, or larger than $\frac{1}{2}$?
$\frac{99}{100} + \frac{100}{101}$: Is the answer smaller than, equal to, or larger than 2?

Numeracy Project materials (see www.nzmaths.co.nz/numeracy/project_material.htm)

- *Book 8: Teaching Number Sense and Algebraic Thinking*
  Advanced multiplicative teaching ideas such as:
  Estimating with Fractions, page 15
  Fractions, page 16
  Equivalent Fractions, page 16.

Extension
Building on multiplication concepts of doubling and halving:

- *Book 6: Teaching Multiplication and Division*
  Cut and Paste, page 25
- *Book 4: Teaching Number Knowledge*
  Who Gets More?, page 20 (comparing the size of fractional numbers)

Pages 16–19: Fishy Fractions

Achievement Objectives

- explain the meaning of the digits in decimal numbers with up to 3 decimal places (Number, level 3)
- express a fraction as a decimal, and vice versa (Number, Level 4)
- make sensible estimates and check the reasonableness of answers (Number, levels 3–4)
- find fractions equivalent to one given (Number, level 4)
- effectively plan mathematical exploration (Mathematical Processes, problem solving, levels 3–4)

AC
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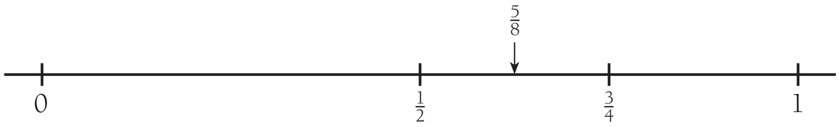
Number Framework Links

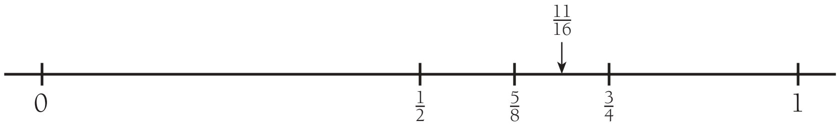
To attempt these activities successfully, students will need to be using multiplicative strategies. Therefore, they will need to be using advanced additive strategies (stage 6) or higher for multiplication and division.

Activities

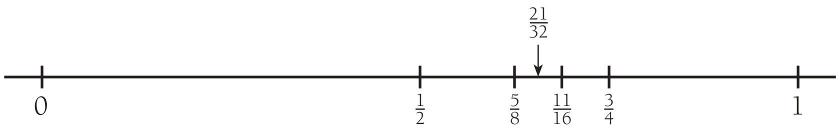
Students often struggle to find a fraction between two fractions if the fractions are close in size but have different denominators. It is an important idea that between any two fractions there is an infinite number of other fractions. For example:



$\frac{5}{8}$ is a fraction that lies between $\frac{1}{2}$ and $\frac{3}{4}$.



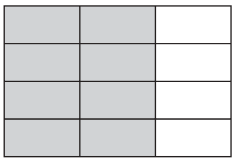
$\frac{11}{16}$ is a fraction that lies between $\frac{5}{8}$ and $\frac{3}{4}$.



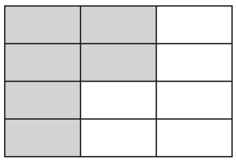
$\frac{21}{32}$ is a fraction that lies between $\frac{5}{8}$ and $\frac{11}{16}$.

The students need to be able to create equivalent fractions that have different denominators from the original fraction in order to find fractions between two fractions. For example, to find a fraction between $\frac{3}{4}$ and $\frac{4}{5}$, both fractions could be converted to equivalent fractions with the same denominator. $4 \times 5 = 20$ is the obvious choice because $\frac{3}{4} = \frac{15}{20}$ and $\frac{4}{5} = \frac{16}{20}$.

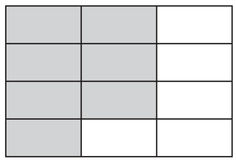
The students are likely to use the part–whole biscuit diagrams as a guide in finding the fractions in between. For example, to find a fraction between $\frac{2}{3}$ and $\frac{1}{2}$, the students might notice that one fraction is $\frac{8}{12}$ of a biscuit and the other is $\frac{6}{12}$. So $\frac{7}{12}$ is in between.



$\frac{2}{3}$ (or $\frac{8}{12}$)



$\frac{1}{2}$ (or $\frac{6}{12}$)



$\frac{7}{12}$

On pages **18–19**, Charu's method of finding a fraction between two fractions involves converting both fractions to decimals. This is similar to the equivalent fractions method in that each fraction is converted to a common base. With decimals, the common bases are tenths, hundredths, thousandths, and so on. Fractions can also be converted to percentages, where the common base is hundredths. It is important that students have experience in converting fractions to decimals and percentages and vice versa because this skill is very important in solving more complex operations. Percentages are often used to make comparisons where the bases are different, for example, comparing basketball shooters who take different numbers of shots.

Both Chris and Hannah use equivalent fractions. In either case, the fractions can be expressed as twelfths. Between $\frac{8}{12}$ and $\frac{9}{12}$, there exists an infinite number of hypothetical fractions like $\frac{8\frac{1}{4}}{12}$, $\frac{8\frac{1}{2}}{12}$, $\frac{8\frac{3}{4}}{12}$, and so on, and these can be converted into equivalent fractions such as $\frac{33}{48}$, $\frac{17}{24}$, $\frac{35}{48}$, and so on. Hannah's method also uses averages. Both Hannah and Chris find the midpoints of the numerators, but Hannah does this by adding the fractions and then dividing by 2.

The students can check that Vaitoa's method works by trying lots of possibilities. The method can also be proved algebraically, but not by students at this level. Vaitoa's method is based on finding the midpoints (averages) of the numerators and the denominators. To find a fraction between $\frac{2}{3}$ and $\frac{5}{6}$, he would find the midpoint between 2 and 5 (that is, $3\frac{1}{2}$) and between 3 and 6 (that is, $4\frac{1}{2}$). The fraction $\frac{3\frac{1}{2}}{4\frac{1}{2}} = \frac{7}{9}$ will lie between $\frac{2}{3}$ and $\frac{5}{6}$.

Question **3** is useful for assessing whether the students are able to apply the strategies to find fractions between fractions. Look for the students to change $2\frac{3}{4}$ and $2\frac{7}{8}$ into improper fractions or to just operate on $\frac{3}{4}$ and $\frac{7}{8}$, knowing that the fraction between will also be between 2 and 3.

Extension

Connect the concept of "betweenness" of fractions to addition and subtraction problems. For example: "*$\frac{1}{2}$ is added to a fraction. The answer is between $\frac{2}{3}$ and $\frac{3}{4}$. What might the fraction be?*" The students should use reverse thinking to realise that the fraction must be between $\frac{1}{6}$ and $\frac{1}{4}$, and they need to understand that an infinite number of fractions will work. Mathematically, this information can be represented using two inequalities: $\frac{1}{6} < n < \frac{1}{4}$, where n represents the possible fraction.

Pages 20–21: Tile the Town, Tiny!

Achievement Objectives

- write and solve problems which involve whole numbers and decimals and which require a choice of one or more of the four arithmetic operations (Number, levels 3–4)
- make sensible estimates and check the reasonableness of answers (Number, level 3)
- use their own language, and mathematical language and diagrams, to explain mathematical ideas (Mathematical Processes, communicating mathematical ideas, levels 3–4)

AC
EA
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Number Framework Links

This activity is aimed at students who are able to apply multiplicative understanding to place value. Use it to encourage transition from advanced additive strategies (stage 6) to advanced multiplicative strategies (stage 7). You can also use this activity with students who are already using advanced multiplicative strategies (stage 7).

Activity

This activity involves estimating and calculating multiplication problems using multiples of 10. The students need to be able to recall groupings of 10 within numbers and round whole numbers.

Many students in this transition struggle with understanding the importance of unit size in multiplication situations. This is further exacerbated by students rote learning rules such as "adding a zero" when multiplying by 10. The rule "add a zero" to multiply by 10 is conceptually inaccurate. Mathematically, adding 0 to a number doesn't change it: 12 + 0 = 12.

The students need to be able to apply basic multiplication facts, such as 3 x 6 = 18, to other multiplication problems. For example:

30 x 60 = 18 x 10 x 10
= 1 800 (18 hundreds)

6 000 x 30 = 18 x 1 000 x 10
= 180 000 (18 ten-thousands)

300 x 0.6 = 18 x 100 x 0.1
= 180 (18 tens)

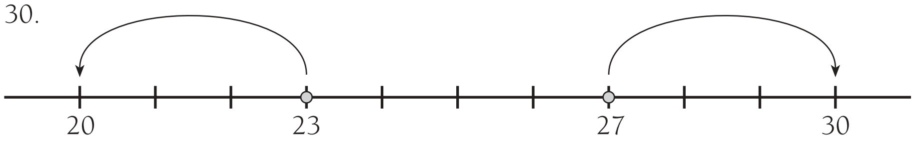
The first part of this activity is aimed at developing this kind of reasoning by having the students partition arrays. The scenario set up, using units of 10 and 100, encourages the students to consider the unit sizes of the numbers involved. For example, 30 x 120 = 3 x 12 x 10 x 10, which is 36 hundreds:

3 tens

100	100	100	100	100	100	100	100	100	100	100	100
100	100	100	100	100	100	100	100	100	100	100	100
100	100	100	100	100	100	100	100	100	100	100	100

12 tens

Question **4** requires the students to apply rounding to multiplicative situations to help them make reasonable estimates. Number lines can be useful for visualising what multiple of 10 or 100 a number is closest to. Question **4i** gives the dimensions of a rectangle as 27 cm x 23 cm. Drawing a number line that contains these two numbers shows that 23 is closer to 20, while 27 is closer to 30.



This means that the area of the 27 cm x 23 cm rectangle can be estimated by finding the area of a 20 cm x 30 cm rectangle. The students may like to draw these rectangles on grid paper and view their areas to see that this makes sense spatially as well.

Question **5** deals with some advanced ideas about the effect of rounding and which strategy is most accurate in which situations. It all depends on the numbers.

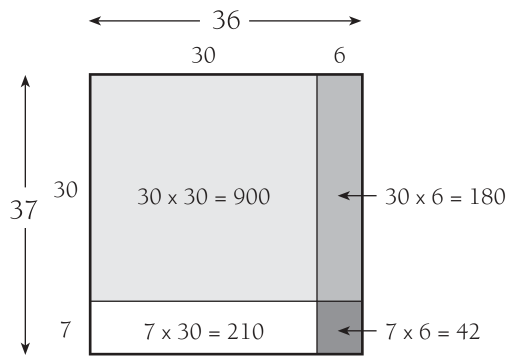
Rounding both dimensions of a rectangle up or down produces variable inaccuracy that is proportional. For example, for a 38 cm x 29 cm rectangle, the estimate of 40 cm x 30 cm = 1 200 cm^2 is only 98 cm^2 away from the actual area, an error of only about 9%. But for a 35 cm x 26 cm rectangle, the rounded estimate is 290 cm^2 away, an error of 32%.

If both dimensions are mid-range between multiples of 10, it is usually more accurate to round one up and one down. Consider a 45 cm x 35 cm rectangle: rounding both numbers up gives an estimate of 2 000 cm^2 (an error of 425 cm^2 or 27%) and rounding both numbers down gives an estimate of 1 200 cm^2 (an error of 375 cm^2 or 24%). Rounding one up and one down gives an estimate of 1 500 cm^2 or 1 600 cm^2 (an error of 75 cm^2 or 25 cm^2). In this case, rounding the bigger factor, 45, down and the smaller factor, 35, up produces the most accurate estimate.

The halfway method described by Tiny Tim in the second example in question **4** involves approximating the area of a rectangle to the area of a square. For example, to estimate the area of a 45 cm x 35 cm rectangle, Tim will use a 40 cm x 40 cm square (45 + 35 = 80. Half of 80 is 40.) In this case, the estimate is very accurate. However, if the dimensions of the rectangle are very different (as in a long skinny rectangle), the method is very inaccurate. Using this method, the area of a 45 cm x 5 cm rectangle (225 cm^2) would be estimated by finding the area of a 25 cm x 25 cm square (625 cm^2).

Extension

Students can learn to multiply large numbers by first splitting factors additively using the distributive property. For example, to calculate 36 x 37, both factors might be split: 36 = 30 + 6, and 37 = 30 + 7. The product is shown in this array:

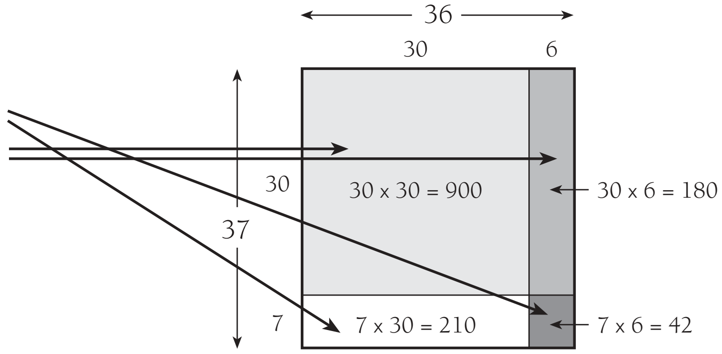


So 36 x 37 = 900 + 180 + 210 + 42.

More about this kind of partitioning can be found on page 37 of *Book 6: Teaching Multiplication and Division* in the Numeracy Project series.

As a further extension, the students could explore the links between the partial products in written algorithms and the parts of the area diagram. For example, 36 x 37 can be solved in written form as:

```
  36
x 37
 252
1080
1332
```



You can extend this application of distributive property by asking the students to work out the answers to problems such as 347 x 28 and 4.6 x 4.5.

Pages 22–24: Wheeling and Dealing

Achievement Objectives

- recall the basic multiplication facts (Number, level 3)
- describe in words, rules for continuing number and spatial sequential patterns (Algebra, level 3)
- use a rule to make predictions (Algebra, level 4)

AC
EA
AA
AM
AP

Number Framework Links

Use these activities to:

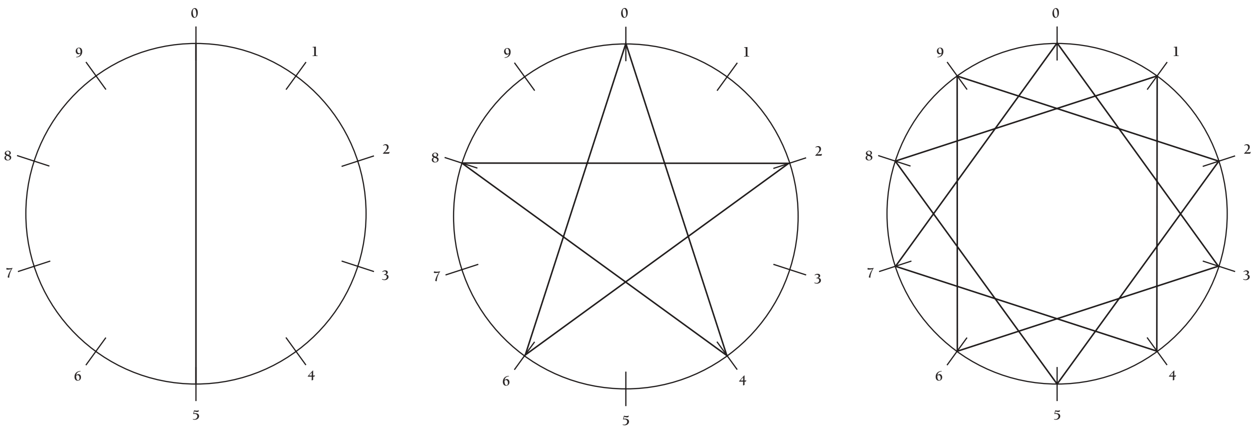
- encourage the use of advanced multiplicative strategies (stage 7) in the domain of multiplication and division
- give students practice in recalling and using basic multiplication facts.

Activity One

This activity is about divisibility: whether a number will divide evenly into another with no remainder. Knowledge of divisibility is very important at the advanced multiplicative stage because it allows students to identify common factors, which is the key to proportional reasoning. For example, to identify that $\frac{48}{64}$ is equivalent to $\frac{3}{4}$, a student must spot that both 48 and 64 are divisible by 2, 4, 8, and 16.

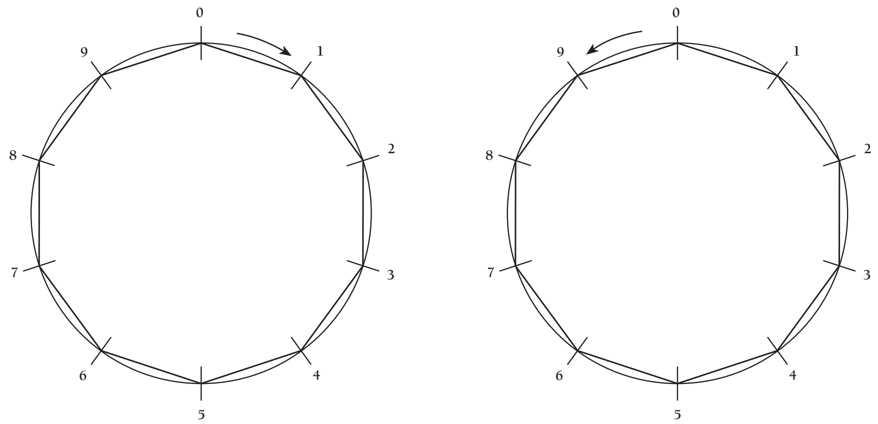
The students need to be aware of the connection between factors and multiples. Multiples are the answers to a given set of multiplication facts. For example, 4, 8, 12, 16, and so on are the multiples of 4. They are numbers that come up when you skip-count in fours from 0. Factors are numbers that multiply together to give a certain number. For example, 4 and 6 are factors of 24 because 4 x 6 = 24. The connection between factors and multiples is this: if 24 is a multiple of 4, then 4 must be one of its factors. Also, this means that 24 must be divisible by 4.

Using a digit wheel to graph the ones digits of multiples produces interesting patterns and can, for some numbers, indicate criteria for being or not being in a set of multiples. For example, digit wheel patterns of the multiples of 5, 4, and 3 look like this:



The digit wheel pattern of multiples of 5 shows that for a number to be a multiple of 5, it must have either 0 or 5 as its ones digit. This is the divisibility test for 5. To be a multiple of 4, the number must have an even number as its ones digit. This is not the definitive rule for divisibility for 4 because there are some even numbers that are not multiples of 4, like 18 and 34. However, it does eliminate all the odd numbers from being multiples of 4. The digit wheel pattern for the multiples of 3 shows that the ones place gives no clue as to whether or not any number is divisible by 3. Every digit is visited on this wheel.

Some multiples form the same pattern on the digit wheel but visit the digits in reverse order. For example, the digit wheels of 1 and 9 both produce this pattern:



Multiples of 2 and 8, 3 and 7, and 4 and 6 also have similar digit wheel patterns. Note how these pairs of matching numbers add up to 10. Question **3b** is set to see if the students can generalise that no matter how many tens, hundreds, and so on are added to the base multiple, the pattern in the ones digit will remain the same. For example, the pattern for multiples of 6 will be the same as that for the multiples of 16, 26, 186, 506, 7 546, and so on. That's because multiplying tens, hundreds, and so on does not alter the ones digit. We need to look further than the ones digit when testing for divisibility by all digits except 1, and 5.

Activity Two

This activity considers a test for divisibility by 3. The test is based on the fact that every 10 is divisible by 3 with 1 left over. Likewise, every hundred, thousand, ten thousand, and so on is also divisible by 3 with 1 left over.

Any whole number is made up of combinations of these powers of 10. For example, 795 is made up of 7 hundreds, 9 tens, and 5 ones. If 1 is taken from each ten and hundred, what is left will be divisible by 3. The only concern then is whether the collection of ones is divisible by 3. 795 has 7 + 9 + 5 = 21 extra ones. 21 is divisible by 3, so 795 must be divisible by 3.

The same principle can be used to test for divisibility by 9 because every ten, hundred, thousand, and so on is divisible by 9 with 1 left over. So for 795, the ones can be collected in the same way, but they must add to a number that is divisible by 9. 21 is not divisible by 9, so 795 is not divisible by 9.

Refer to pages 19–33 of *Book 6: Teaching Multiplication and Division* in the Numeracy Project series for lesson notes on teaching divisibility by 9 and 3.

Extension

The students can explore how the divisibility tests for 3 and 9 can be used to work out division answers and remainders. In the case of 795:

- Each 100 contains 33 threes with 1 left over, that is, 7 x 33 = 231 threes with a remainder of 7 ones.
- Each 10 contains 3 threes with 1 left over, that is, 9 x 3 = 27 threes with a remainder of 9 ones.
- The extra 7 + 9 + 5 = 21 ones make another 7 threes.
- This gives a total of 231 + 27 + 7 = 265 threes. So 795 ÷ 3 = 265.

By similar reasoning, 795 divided by 9 must be:

- 77 nines from the hundreds with 7 ones remaining
- 9 nines from the tens, with 9 ones remaining.
- From the 7 + 9 + 5 = 21 ones, another 2 nines can be made with 3 remaining.
- The total number of nines is 77 + 9 + 2 = 88, so 795 ÷ 9 = 88 r3.

Note that the answer to a given number divided by 3 is always 3 times larger than the same number divided by 9. In the case of 795, 265 = $88\frac{1}{3}$ x 3 (because the remainder 3 is $\frac{1}{3}$ of 9).

The students can also investigate other divisibility rules, such as:

- If the last two digits of any whole number form a 2-digit number that is divisible by 4, then the number is divisible by 4. For example, 78 536 is divisible by 4 because 36 is divisible by 4. This results because every hundred, thousand, and so on is divisible by 4, for example, 100 ÷ 4 = 25.
- If a number is divisible by 2 (an even number) and it is also divisible by 3, then the number is divisible by 6. For example, 576 is divisible by 3 because 5 + 7 + 6 = 18. 576 is also even, so it must be divisible by 6, that is, 576 ÷ 6 = 96.

Numeracy Project materials (see www.nzmaths.co.nz/numeracy/project_material.htm)

- *Book 8: Teaching Number Sense and Algebraic Thinking*
  Divisibility Tests, page 33
  For later advanced proportional work:
  Highest Common Factors (HCFs), page 39
  Lowest Common Multiples (LCMs), page 40

Copymaster: Stripping Fractions

1					
$\frac{1}{2}$	$\frac{1}{2}$				
$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$			
$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$		
$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	
$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Copymaster: Stripping Fractions

1

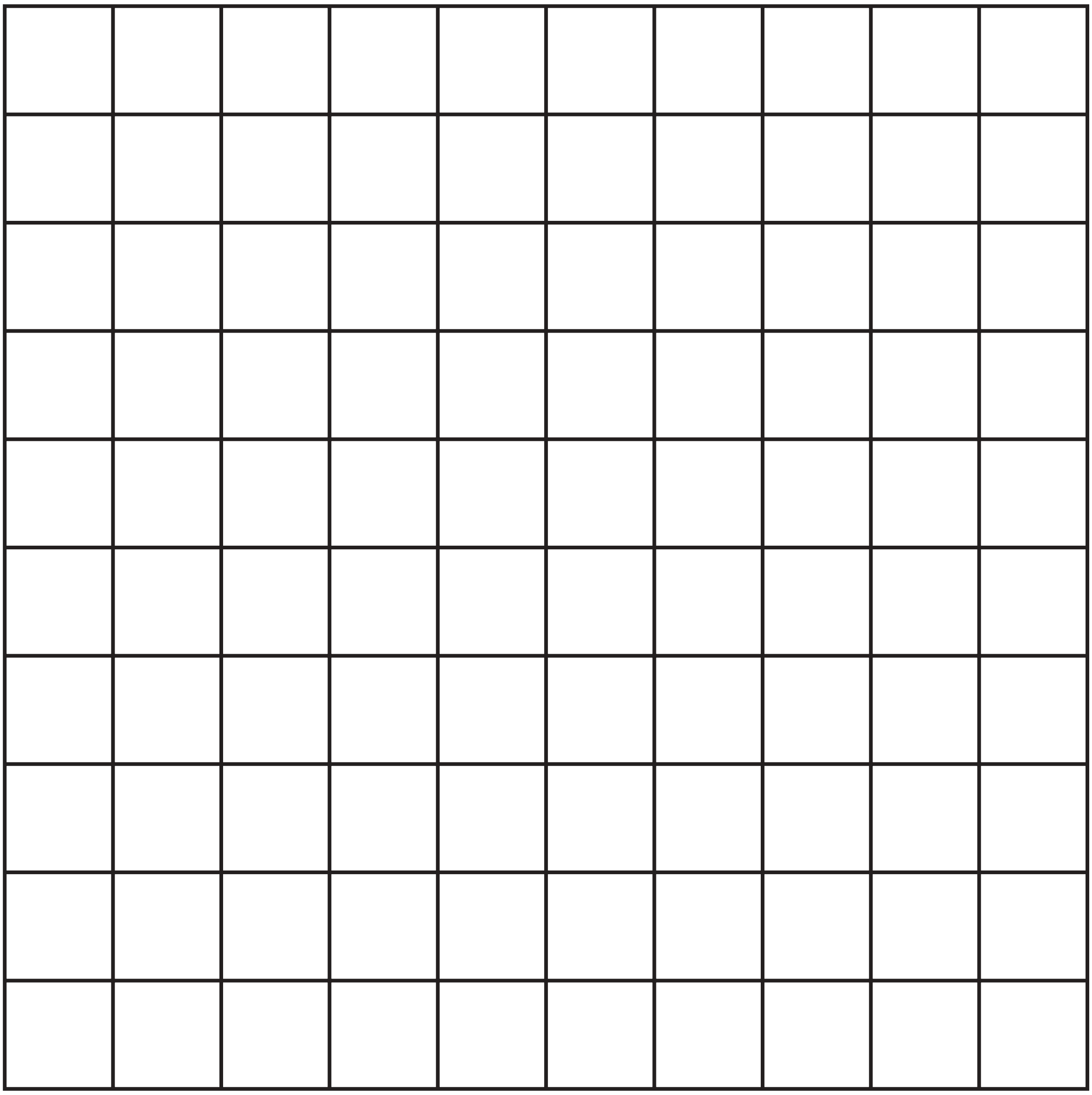
$\frac{1}{7}$ $\frac{1}{7}$ $\frac{1}{7}$ $\frac{1}{7}$ $\frac{1}{7}$ $\frac{1}{7}$ $\frac{1}{7}$

$\frac{1}{8}$ $\frac{1}{8}$ $\frac{1}{8}$ $\frac{1}{8}$ $\frac{1}{8}$ $\frac{1}{8}$ $\frac{1}{8}$ $\frac{1}{8}$

$\frac{1}{9}$ $\frac{1}{9}$ $\frac{1}{9}$ $\frac{1}{9}$ $\frac{1}{9}$ $\frac{1}{9}$ $\frac{1}{9}$ $\frac{1}{9}$ $\frac{1}{9}$

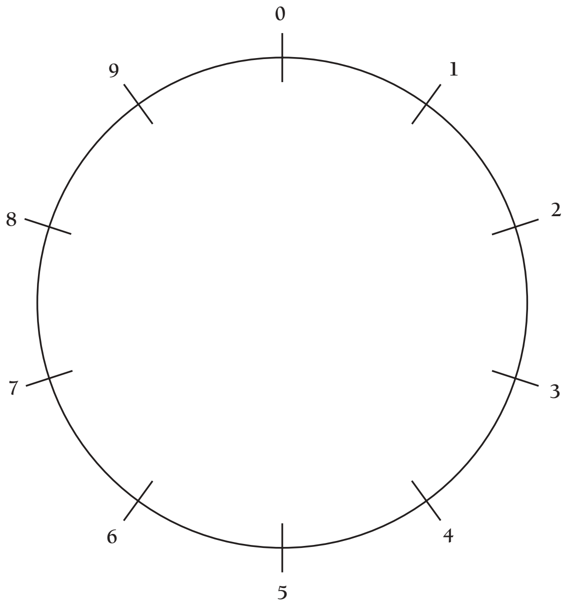
$\frac{1}{10}$ $\frac{1}{10}$ $\frac{1}{10}$ $\frac{1}{10}$ $\frac{1}{10}$ $\frac{1}{10}$ $\frac{1}{10}$ $\frac{1}{10}$ $\frac{1}{10}$ $\frac{1}{10}$

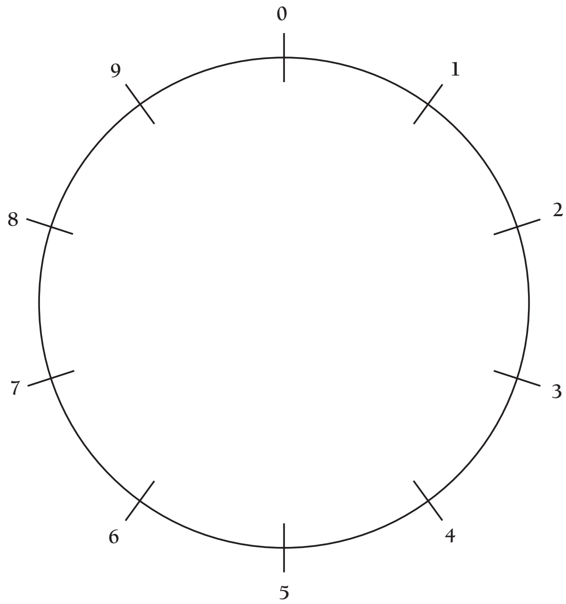
Copymaster: Tile the Town, Tiny!

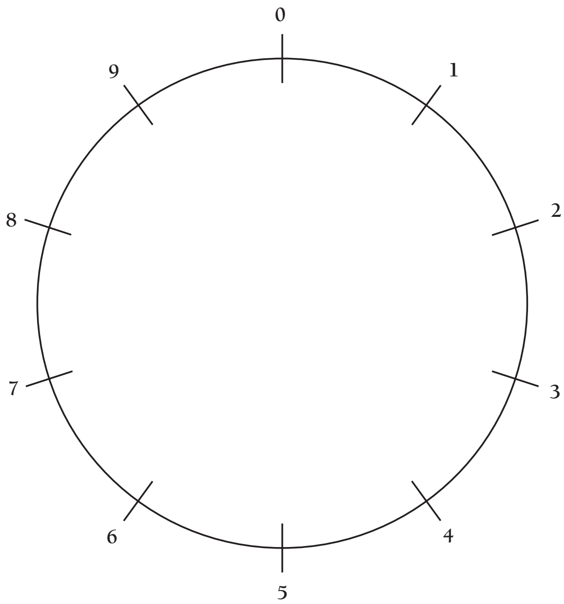


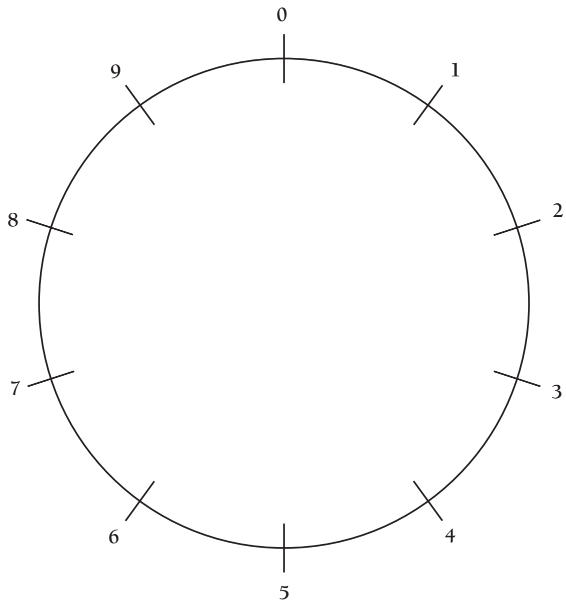
100-tile square

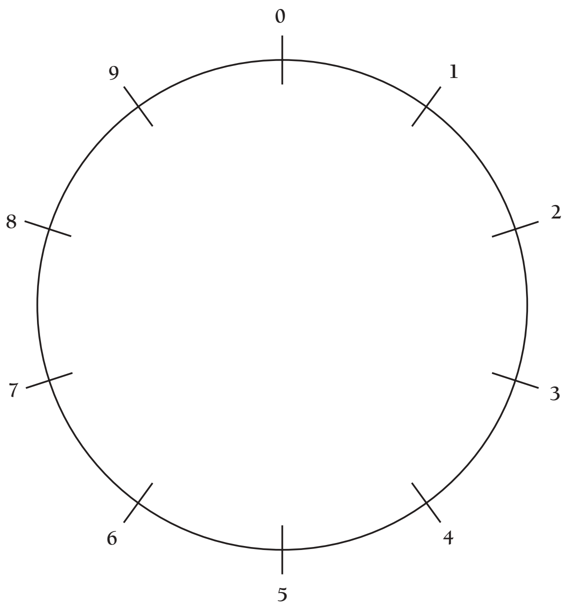
Copymaster: Wheeling and Dealing

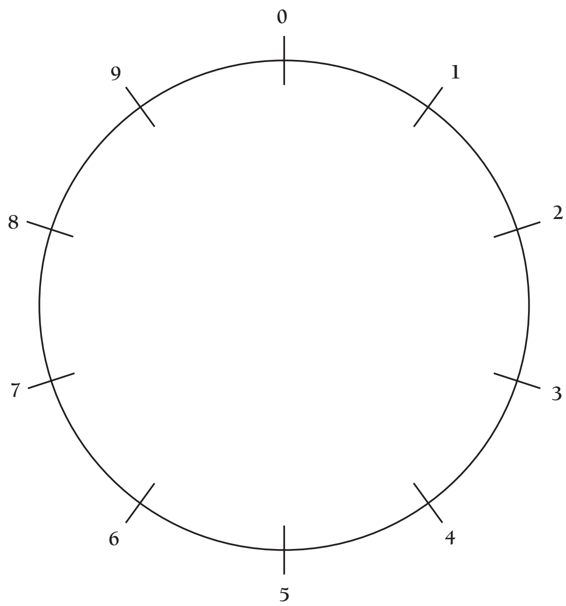












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