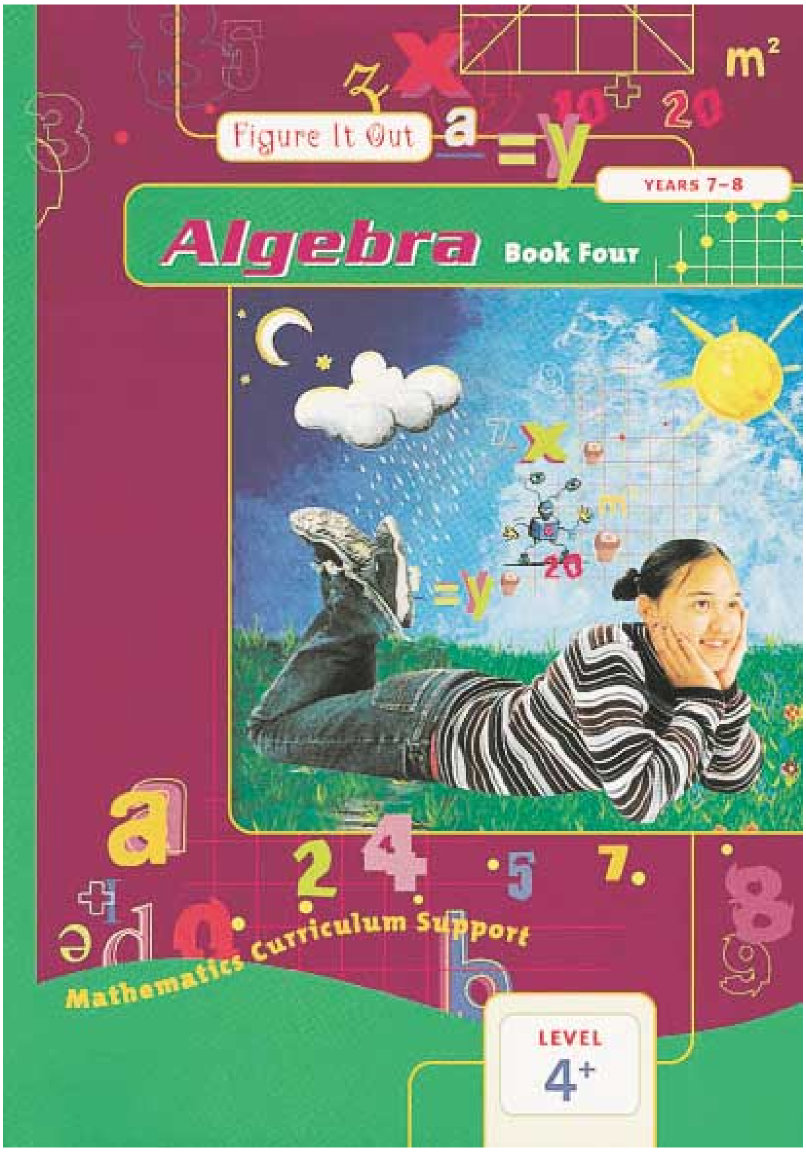
Answers and Teachers' Notes



MINISTRY OF EDUCATION
Te Tāhuhu o te Mātauranga

CONTENTS

The books for years 7–8 in the Figure It Out series are issued by the Ministry of Education to provide support material for use in New Zealand year 7–8 classrooms. The books have been developed and trialled by classroom teachers and mathematics educators and follow on from the successful series for levels 2–4 in primary schools.

Student books

The student books in the series are divided into three curriculum levels: levels 2–3 (linking material), level 4, and level 4+ (extension material). All the books are aimed at year 7–8 students in terms of context and presentation.

The following books are included in the series:
Number (two linking, three level 4, one level 4+, distributed in November 2002)
Number Sense (one linking, one level 4, distributed in April 2003)
Algebra (one linking, two level 4, one level 4+, distributed in August 2003)
Geometry (one level 4, one level 4+, distributed in term 1 2004)
Measurement (one level 4, one level 4+, distributed in term 1 2004)
Statistics (one level 4, one level 4+ distributed in term 1 2004)
Themes: *Disasters Strike!, Getting Around* (level 4–4+, distributed in August 2003)

The activities in the student books are set in meaningful contexts, including real-life and imaginary scenarios. The books have been written for New Zealand students, and the contexts reflect their ethnic and cultural diversity and life experiences that are meaningful to students aged 11–13 years. The activities can be used as the focus for teacher-led lessons, as independent bookwork, or as the catalyst for problem solving in groups.

Answers and Teachers' Notes

The Answers section of the *Answers and Teachers' Notes* that accompany each of the student books includes full answers and explanatory notes. Students can use them for self-marking, or you can use them for teacher-directed marking. The teachers' notes for each activity, game, or investigation include relevant achievement objectives, comment on mathematical ideas, processes, and principles, and suggestions on teaching approaches. The *Answers and Teachers' Notes* are also available on Te Kete Ipurangi (TKI) at www.tki.org.nz/r/maths/curriculum/figure

Using Figure It Out in your classroom

Where applicable, each page starts with a list of equipment that the students will need to do the activities. Encourage the students to be responsible for collecting the equipment they need and returning it at the end of the session.

Many of the activities suggest different ways of recording the solution to a problem. Encourage your students to write down as much as they can about how they did investigations or found solutions, including drawing diagrams. Discussion and oral presentation of answers is encouraged in many activities, and you may wish to ask the students to do this even where the suggested instruction is to write down the answer.

> The ability to communicate findings and explanations, and the ability to work satisfactorily in team projects, have also been highlighted as important outcomes for education. Mathematics education provides many opportunities for students to develop communication skills and to participate in collaborative problem-solving situations.
>
> *Mathematics in the New Zealand Curriculum*, page 7

Students will have various ways of solving problems or presenting the process they have used and the solution. You should acknowledge successful ways of solving questions or problems, and where more effective or efficient processes can be used, encourage the students to consider other ways of solving a particular problem.

Answers

Algebra: Book Four

Page 1 Square Number Differences

ACTIVITY

1. **a.** $3^2 - 2^2 = 5$ and $4^2 - 3^2 = 7$

   b. The difference between consecutive square numbers is always odd. The difference is the sum of the two numbers that are squared.

   c. 19 + 18 = 37, so $19^2 - 18^2 = 37$

2. **a.** $4^2 - 2^2 = 12$ and $5^2 - 3^2 = 16$

   b. The difference between alternate square numbers is always even; it is twice the sum of the two numbers that are squared.

   c. 2 x (13 + 11) = 48, so $13^2 - 11^2 = 48$

3. **a.** $5^2 - 2^2 = 21$ and $6^2 - 3^2 = 27$

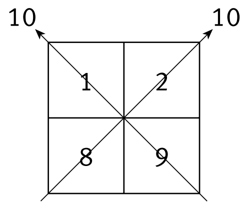
   b. The difference is always odd; it can be worked out by trebling (multiplying by 3) the sum of the two numbers that are squared.

   c. 3 x (17 + 14) = 93, so $17^2 - 14^2 = 93$

Pages 2–3 Calendars and Short Cuts

ACTIVITY

1. **a.** Answers will vary. The simplest short cut is to first add the diagonal numbers and then add the two totals.



So (9 + 1) + (8 + 2) = 10 + 10, which is 20. Another short cut is based on the fact that the sum of the numbers on the two diagonals is always equal, so this is the same as adding the numbers in one diagonal and then doubling the result: 2 x (9 + 1) = 20.

   b. **i.** (10 + 2) + (9 + 3) = 24 or 2 x (10 + 2) = 24

   ii. (19 + 11) + (18 + 12) = 60 or 2 x (19 + 11) = 60

   iii. (23 + 15) + (22 + 16) = 76 or 2 x (23 + 15) = 76

   iv. (28 + 20) + (27 + 21) = 96 or 2 x (28 + 20) = 96

2. **a.** There are several sensible short cuts. Two are shown here. (Another short cut is the one that Anna uses in **b.**)

Short cut 1:

3	4	5
10	11	12
17	18	19

3 + 4 + 5 = 12

(3 + 7) + (4 + 7) + (5 + 7)
= (3 + 4 + 5) + 3 x 7

(3 + 7 + 7) + (4 + 7 + 7) + (5 + 7 + 7)
= (3 + 4 + 5) + 6 x 7

So the short cut is
3 x (3 + 4 + 5) + 9 x 7 = 36 + 63
= 99.

Short cut 2:

3	4	5
10	11	12
17	18	19

3 + 10 + 17 = 30
4 + 11 + 18 = 33
5 + 12 + 19 = 36

So the short cut is 30 + 33 + 36, which is the same as 3 x 33 = 99.

b. Explanations may vary. All the other numbers in the grid can be written in relation to the middle number:

11 – 8 = 3	11 – 7 = 4	11 – 6 = 5
11 – 1 = 10	11	11 + 1 = 12
11 + 6 = 17	11 + 7 = 18	11 + 8 = 19

The sum of the numbers in the two shaded corner boxes is (11 – 8) + (11 + 8) = 11 + 11
= 2 x 11.

There are three other pairs of numbers that work like this. They are:
(11 – 7) + (11 + 7) = 2 x 11
(11 – 6) + (11 + 6) = 2 x 11
(11 – 1) + (11 + 1) = 2 x 11.

The sum of these 4 sets of numbers is 4 x 2 x 11, which is the same as 8 x 11. When the middle number is added to this, the sum of the 9 numbers is 8 x 11 + 1 x 11 = 9 x 11. The rule can be expressed as "the number of numbers multiplied by the middle number".

c. **i.** 9 x 9 = 81

ii. 9 x 12 = 108

iii. 9 x 20 = 180

iv. 9 x 21 = 189

3. **a.** Angus is correct. 1 + 2 + 3 + 4 + 8 + 9 + 10 + 11 + 15 + 16 + 17 + 18 + 22 + 23 + 24 + 25 = 208

b. Explanations may vary. Angus starts by finding the middle number in the square of 16 numbers. He uses two corner numbers, 1 and 25:
$\frac{1 + 25}{2} = \frac{26}{2}$, or 13.
(You could also use 4 and 22 because the average of two diagonally opposed numbers is always the same as the average of their opposite pair.) As with the 9 numbers in the 3 by 3 array of numbers, the total of the 16 numbers can be found from the rule "the number of numbers multiplied by the middle number" or, in Angus's case, "the middle number multiplied by the number of numbers". So the short cut is 16 x 13 = 208 or 13 x 16 = 208.

c. **i.** 21 x 13 = 273

ii. 5 x 16 = 80

iii. 31 x 16 = 496

4. **a.** $\frac{1 + 45}{2}$ x 25 = 23 x 25
= 575
(1 + 2 + 3 + 4 + 5 ... + 43 + 44 + 45 = 575)

b. Explanations may vary. As with the previous examples, the rule is: the number of numbers multiplied by the middle number. There are 25 numbers, and Jacob finds the middle number by adding up the smallest and the largest number and dividing the result by 2. The middle number is 23, so the total of the numbers is 25 x 23 = 575.

c. Possible short cuts include:

i. 63 x 44 = 2 772 or $\frac{1 + 87}{2}$ x 63 = 2 772

ii. 35 x 45 = 1 575 or $\frac{22 + 68}{2}$ x 35 = 1 575

iii. 100 x 50.5 = 5 050 or
$\frac{1 + 100}{2}$ x 100 = 5 050

Pages 4–5 Number Juggling

ACTIVITY

Challenge One

Each starting number produces the answer 4 (in step 4). The starting number is disguised by two multiplication steps and removed by division to give the answer of 4.

Step 1 Starting number	Step 2 Multiply by 2	Step 3 Square	Step 4 Divide by square of starting number
1	1 x 2 = 2	2^2 = 4	4 ÷ 1^2 = 4
2	2 x 2 = 4	4^2 = 16	16 ÷ 2^2 = 4
3	3 x 2 = 6	6^2 = 36	36 ÷ 3^2 = 4
4	4 x 2 = 8	8^2 = 64	64 ÷ 4^2 = 4
5	5 x 2 = 10	10^2 = 100	100 ÷ 5^2 = 4
10	10 x 2 = 20	20^2 = 400	400 ÷ 10^2 = 4
30	30 x 2 = 60	60^2 = 3 600	3 600 ÷ 30^2 = 4

Challenge Two

Each starting number produces the answer 9 (step 5). The starting number is disguised by a mixture of subtraction, multiplication, and addition and then removed by multiplication and subtraction to give the answer of 9.

Step 1	Step 2	Step 3	Step 4	Step 5
Starting number	Subtract 3	Square	Add 6 times the starting number	Subtract square of starting number
1	$1 - 3 = {}^{-}2$	${}^{-}2^2 = 4$	$4 + 6 \times 1 = 10$	$10 - 1^2 = 9$
2	$2 - 3 = {}^{-}1$	${}^{-}1^2 = 1$	$1 + 6 \times 2 = 13$	$13 - 2^2 = 9$
3	$3 - 3 = 0$	$0^2 = 0$	$0 + 6 \times 3 = 18$	$18 - 3^2 = 9$
4	$4 - 3 = 1$	$1^2 = 1$	$1 + 6 \times 4 = 25$	$25 - 4^2 = 9$
5	$5 - 3 = 2$	$2^2 = 4$	$4 + 6 \times 5 = 34$	$34 - 5^2 = 9$
6	$6 - 3 = 3$	$3^2 = 9$	$9 + 6 \times 6 = 45$	$45 - 6^2 = 9$
7	$7 - 3 = 4$	$4^2 = 16$	$16 + 6 \times 7 = 58$	$58 - 7^2 = 9$
10	$10 - 3 = 7$	$7^2 = 49$	$49 + 6 \times 10 = 109$	$109 - 10^2 = 9$
30	$30 - 3 = 27$	$27^2 = 729$	$729 + 6 \times 30 = 909$	$909 - 30^2 = 9$

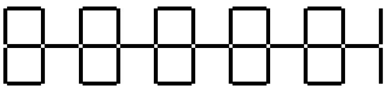
INVESTIGATION

Challenges will vary. In all cases, the original number will be disguised in some way (for example, by multiplication) and then removed in the course of the final step or steps (for example, by division).

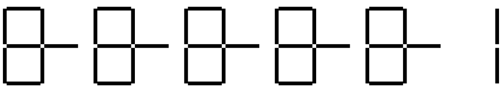
Pages 6–7 Initials Logo

ACTIVITY

1. a. This design with 5 joined logos has 42 sticks, so Evalesi is correct.



b. The logo design can be divided into 6 parts. 5 parts are identical and have 8 sticks each. The last part has 2 sticks. So there are 5 x 8 + 2 sticks altogether.



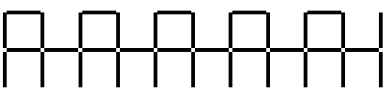
5 sets of 8 sticks 2 sticks

c. 100 x 8 + 2 = 802 sticks

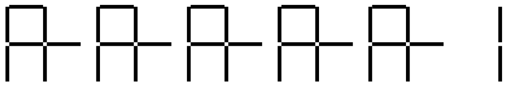
d.

Number of joined logos	Number of sticks
5	5 x 8 + 2 = 42
6	6 x 8 + 2 = 50
20	20 x 8 + 2 = 162
94	94 x 8 + 2 = 754
256	256 x 8 + 2 = 2 050

2. a. This design with 5 joined logos has 37 sticks, so Arnon is correct.



b. The logo design can be divided into 6 parts. 5 parts are identical and have 7 sticks each. The last part has 2 sticks. So there are 5 x 7 + 2 sticks altogether.



5 sets of 7 sticks 2 sticks

c. 100 x 7 + 2 = 702 sticks

d.

Number of joined logos	Number of sticks
5	5 x 7 + 2 = 37
7	7 x 7 + 2 = 51
36	36 x 7 + 2 = 254
87	87 x 7 + 2 = 611
109	109 x 7 + 2 = 765

3. a. Using the 5 orange sticks enables Evalesi to see 4 sets of 7 (the normal repeat in the design) or 4 x 7 sticks. The 5 orange sticks are not part of the design and must be removed. So the design has 4 x 7 – 5 = 23 sticks.

b. 6 x 7 – 5 = 37 sticks

c.

Number of joined logos	Number of sticks
3	4 x 7 – 5 = 23
9	10 x 7 – 5 = 65
15	16 x 7 – 5 = 107
47	48 x 7 – 5 = 331
183	184 x 7 – 5 = 1 283

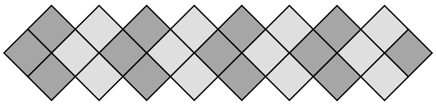
d.

Number of joined logos	Number of sticks
6	44 (7 x 7 – 5)
2 ((16 + 5) ÷ 7 – 1)	16
4 ((30 + 5) ÷ 7 – 1)	30
10 ((72 + 5) ÷ 7 – 1)	72
90 ((632 + 5) ÷ 7 – 1)	632

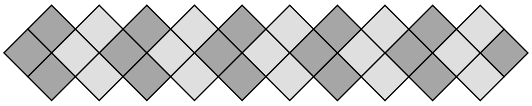
Pages 8–9 Bathroom Tiles

ACTIVITY

1. a.



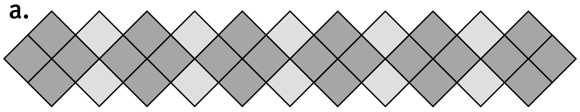
b.



c. i. 30 pink tiles and 30 orange tiles and an extra pink tile

ii. 57 pink tiles and 54 orange tiles and an extra orange tile. (37 sets of 3 tiles gives 19 pink sets and 18 orange sets. The last set of 3 tiles is pink, so the extra tile is orange.)

2. a.



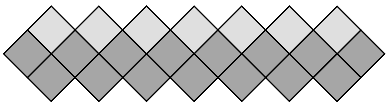
b. i. 112 tiles. (A design with 36 orange tiles has 19 sets of 4 pink tiles and 18 sets of 2 orange tiles. 19 x 4 + 18 x 2 = 112 tiles)

ii. 298 tiles. (50 x 4 pink tiles + 49 x 2 orange tiles = 298 tiles)

c.

Number of orange tiles	Number of pink tiles	Total number of tiles
8	(8 ÷ 2 + 1) x 4 = 20	8 + 20 = 28
20	(20 ÷ 2 + 1) x 4 = 44	20 + 44 = 64
28	(28 ÷ 2 + 1) x 4 = 60	28 + 60 = 88
42	(42 ÷ 2 + 1) x 4 = 88	42 + 88 = 130
156	(156 ÷ 2 + 1) x 4 = 316	156 + 316 = 472

3. a. Bill is correct:



There are 7 orange tiles and 7 x 2 + 1 = 15 purple tiles.

b. For each orange tile, there are 2 purple tiles. There is also an additional purple tile. So for a design with 7 orange tiles, there are 7 x 2 + 1 purple tiles.

c. 87 x 2 + 1 = 175 purple tiles

d.

Number of orange tiles	Number of purple tiles	Total number of tiles
4	4 x 2 + 1 = 9	4 + 9 = 13
8	8 x 2 + 1 = 17	8 + 17 = 25
11	11 x 2 + 1 = 23	11 + 23 = 34
(43 – 1) ÷ 2 = 21	43	21 + 43 = 64
(101 – 1) ÷ 2 = 50	101	50 + 101 = 151

e.

Number of orange tiles	Total number of tiles
(31 – 1) ÷ 3 = 10	31
(52 – 1) ÷ 3 = 17	52
(250 – 1) ÷ 3 = 83	250

The total number of tiles is 3 x the number of orange tiles + 1.

Two possible short cuts for finding the total number of tiles are shown in the tables below. They are based on the fact that in each set of 3 tiles, 1 orange tile is half the number of purple tiles. (The total number of purple tiles also includes the extra purple tile at the end.)

Number of purple tiles	Total number of tiles
11	(3 x 11 – 1) ÷ 2 = 16
15	(3 x 15 – 1) ÷ 2 = 22
37	(3 x 37 – 1) ÷ 2 = 55

Number of purple tiles	Total number of tiles
11	11 + $^1/_2$(11 – 1) = 16
15	15 + $^1/_2$(15 – 1) = 22
37	37 + $^1/_2$(37 – 1) = 55

4. a. Answers may vary. A possible short cut is (8 – 1) x 2 = 14. For 4 orange tiles, there are (4 – 1) x 2 = 6 purple tiles, and for 6 orange tiles, there are (6 – 1) x 2 = 10 purple tiles. So if there are 8 orange tiles, there are (8 – 1) x 2 = 14 purple tiles.

b. (47 – 1) x 2 = 92 purple tiles

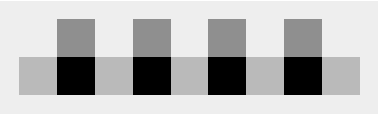
c. Based on the short cut above:

Number of orange tiles	Number of purple tiles	Total number of tiles
11	(11 – 1) x 2 = 20	31
21	(21 – 1) x 2 = 40	61
37	(37 – 1) x 2 = 72	109
(44 ÷ 2) + 1 = 23	44	67
(58 ÷ 2) + 1 = 30	58	88
(198 ÷ 2) + 1 = 100	198	298

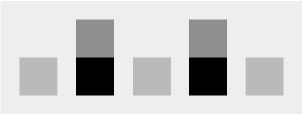
Pages 10–11 Patterns and Designs

ACTIVITY

1. a.

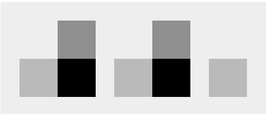


b. i. A possible diagram is:



ii. 28 x 2 + 29 = 85 coloured squares

c. The diagram below shows 2 sets of 3 coloured squares forming an L shape, and 1 additional square.



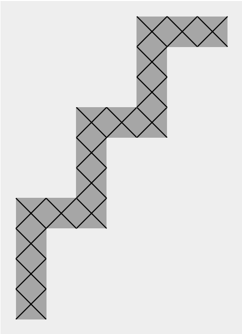
So the number of coloured squares is
2 x 3 + 1 = 7.

d.

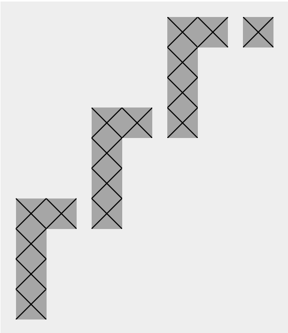
Number of green squares	Total number of coloured squares
5	16 5 x 2 + 6 or 5 x 3 + 1
7	22 7 x 2 + 8 or 7 x 3 + 1
12	37 12 x 2 + 13 or 12 x 3 + 1
37	112 37 x 2 + 38 or 37 x 3 + 1
143	430 143 x 2 + 144 or 143 x 3 + 1

Number of green squares	Total number of coloured squares
8 (25 – 1) ÷ 3	25
10 (31 – 1) ÷ 3	31
17 (52 – 1) ÷ 3	52
28 (85 – 1) ÷ 3	85
33 (100 – 1) ÷ 3	100

2. a.



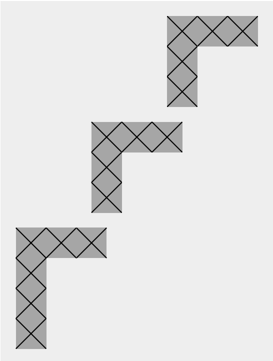
b. Answers may vary. One possible short cut is 7 x 5 + 1 = 36. It works like this for 3 steps:



3 x 5 + 1 = 16

So the pattern with 7 steps has 7 x 5 + 1 = 36 crosses.

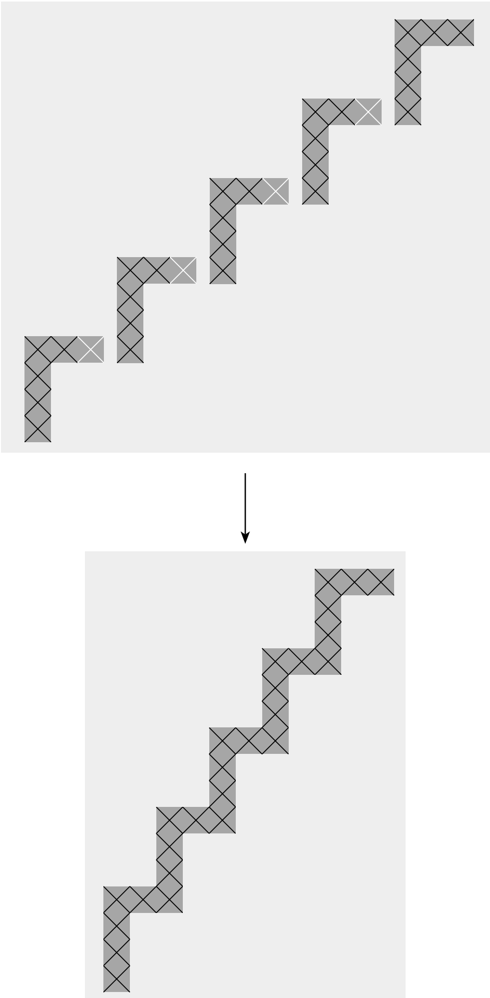
Another short cut for 3 steps is:



1 set of 6 + 2 sets of 5 = 6 + (2 x 5)
= 16

So the pattern with 7 steps has 6 + (6 x 5) = 36 crosses.

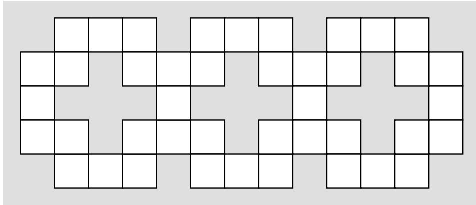
c. Yes, it is correct. Hine included an additional cross (white) in all but the final step, so each step has 6 crosses. The 4 white crosses must be subtracted, so there are 5 x 6 – 4 = 26 crosses altogether.



d. Short cuts for the Your rule column are based on the short cuts suggested in **2b**.

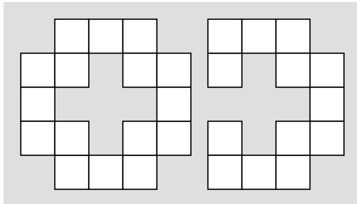
Number of steps	Number of crosses	
	Your rule	Hine's rule
5	26 5 x 5 + 1 or 6 + (4 x 5)	5 x 6 – 4 = 26
10	51 10 x 5 + 1 or 6 + (9 x 5)	10 x 6 – 9 = 51
37	186 37 x 5 + 1 or 6 + (36 x 5)	37 x 6 – 36 = 186
78	391 78 x 5 + 1 or 6 + (77 x 5)	78 x 6 – 77 = 391
100	501 100 x 5 + 1 or 6 + (99 x 5)	100 x 6 – 99 = 501
342	1 711 342 x 5 + 1 or 6 + (341 x 5)	342 x 6 – 341 = 1 711

3. a.



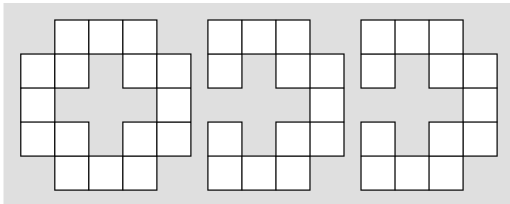
b. Answers may vary. One possible short cut works like this:

2 plus signs



16 + 1 x 13 = 29 cubes

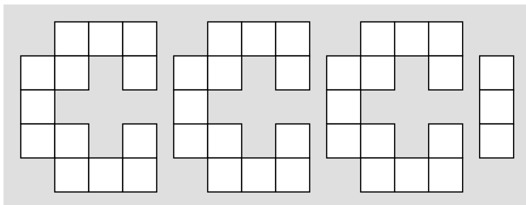
3 plus signs



16 + 2 x 13 = 42 cubes

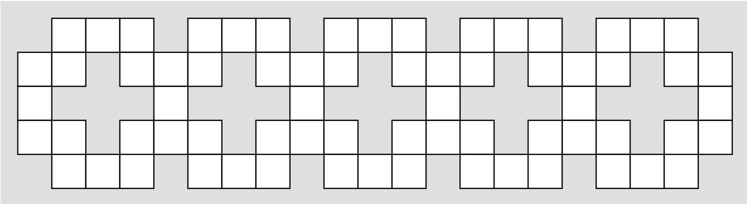
A possible rule is: 16 for the first plus sign + the number of extra plus signs x 13. So the short cut for 5 plus signs is 16 + 4 x 13 = 68.

Another short cut is:



3 sets of 13 cubes + 3 cubes
3 x 13 + 3 = 42 cubes

A possible rule is: 13 for each plus sign plus an extra 3. So the short cut for 5 plus signs is 5 x 13 + 3 = 68.

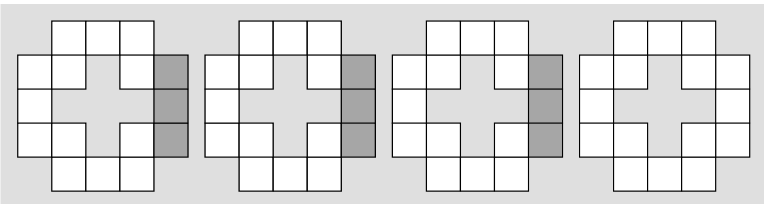
c. 

The 5 plus signs use 68 multilink cubes as predicted by the short cuts 16 + 4 x 13 and 5 x 13 + 3.

d. The calculations are based on the two rules given in **3b**.

Number of crosses	Number of multilink cubes	
3	16 + 2 x 13 = 42	3 x 13 + 3 = 42
7	16 + 6 x 13 = 94	7 x 13 + 3 = 94
10	16 + 9 x 13 = 133	10 x 13 + 3 = 133
100	16 + 99 x 13 = 1 303	100 x 13 + 3 = 1 303
343	16 + 342 x 13 = 4 462	343 x 13 + 3 = 4 462

4. a. When there are 4 plus signs, the first short cut from question **3** gives 16 + 3 x 13 = 55 cubes. Tracey's rule, which allows for extra cubes to be subtracted, gives 4 x 16 – 3 x 3 = 55. Both rules give the answer as 55, so Tracey's rule for 4 plus signs is correct.



4 x 16 – 3 x 3 = 55

b. Tracey's short cut is 8 x 16 – 7 x 3 = 107. Other possible short cuts, such as 16 + 7 x 13 = 107 and 8 x 13 + 3 = 107, should give the same result.

c. 1 000 x 16 – 999 x 3 = 16 000 – 2 997
= 13 003

Page 12 **Alien Critters**

ACTIVITY

1.

Number of critters		Number of asteroids	Minimum number of steps	Minimum number of jumps	Minimum total number of moves
Blue	Red				
1	1	3	2	1	3
2	2	5	4	4	8
3	3	7	6	9	15

2. a. Reuben may have noticed a pattern in the columns for the minimum number of steps (increase by 2) and the minimum number of jumps (square numbers).

Number of critters		Number of asteroids	Minimum number of steps	Minimum number of jumps	Minimum total number of moves
Blue	Red				
1	1	3	**2**	**1**	3
2	2	5	**4**	**4**	8
3	3	7	**6**	**9**	15
4	4	9	**8**	**16**	24

b. A possible rule is: the minimum number of steps is twice the number of blue or red critters.
The minimum number of jumps is the square of the number of blue or red critters.

3. **a.** A possible rule is: the minimum number of moves is the minimum number of steps plus the minimum number of jumps. This can be found using the rules for the number of steps and the number of jumps given for question **2b**.

Alternatively, the second pattern in the table below indicates that the minimum number of moves can be found by multiplying the number of red (or blue) critters by 2 more than the number of red (or blue) critters.

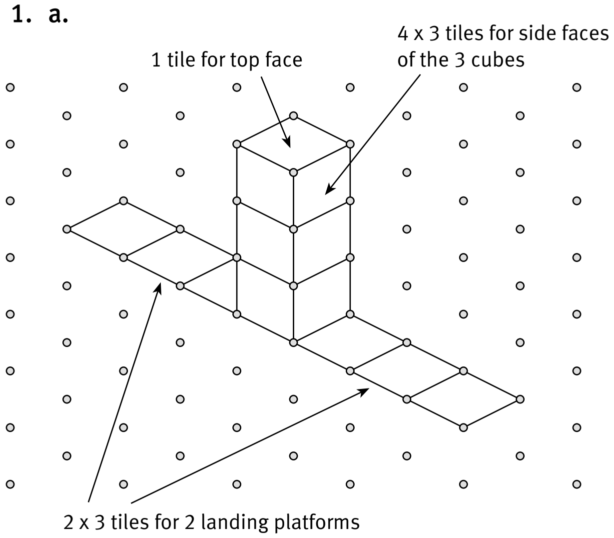
Number of blue or red critters	Pattern	Alternative pattern	Minimum number of moves
1	$2 \times 1 + 1^2 = 3$	$1 \times 3 = 3$	3
2	$2 \times 2 + 2^2 = 8$	$2 \times 4 = 8$	8
3	$2 \times 3 + 3^2 = 15$	$3 \times 5 = 15$	15
4	$2 \times 4 + 4^2 = 24$	$4 \times 6 = 24$	24
5	$2 \times 5 + 5^2 = 35$	$5 \times 7 = 35$	35
6	$2 \times 6 + 6^2 = 48$	$6 \times 8 = 48$	48

b. $2 \times 100 + 100^2 = 10\,200$ moves or $100 \times (100 + 2) = 10\,200$ moves.

Page 13 Tiling Spacecraft

ACTIVITY

1. **a.**



b. $1 + 5 \times 4 + 2 \times 5 = 31$ square tiles. The drawing will have 2 more cubes and 2 more tiles on each landing platform than the one in **a.**

2. In the rule $1 + 4 \times n + 2 \times n$, the letter n stands for any number of cubes that make up the body of the spacecraft. For example, when $n = 5$, the spacecraft looks like 5 stacked cubes. There will be 1 square tile on the top face and 4 square tiles on the side faces of each of the 5 cubes, that is, $1 + 4 \times 5$ tiles. Each of the 2 landing platforms has 5 square tiles, so there are 2×5 square tiles for the landing platforms. So a 5-cube spacecraft has $1 + 4 \times 5 + 2 \times 5$ square tiles.
A 10-cube spacecraft has $1 + 4 \times 10 + 2 \times 10$ square tiles, and a 100-cube spacecraft has $1 + 4 \times 100 + 2 \times 100$ square tiles. In general, a spacecraft with any number (n) of stacked cubes will have $1 + 4n + 2n$ square tiles.

3. **a.**

Craft size (number of cubes)	1	2	3	4	5
Number of square tiles	7	13	19	25	31

b. Fran notices that the number of square tiles is 1 more than 6 times the number of cubes. So for any number (n) of stacked cubes, there are $6 \times n + 1$ or $6n + 1$ square tiles.

4. **a.**

Craft size (number of cubes)	Number of square tiles	
	Jeff's predictions	Fran's predictions
1	$1 + 4 \times 1 + 4 \times 1 = 9$	$8 \times 1 + 1 = 9$ or $4 \times 2 + 1 = 9$
2	$1 + 4 \times 2 + 4 \times 2 = 17$	$8 \times 2 + 1 = 17$ or $4 \times 4 + 1 = 17$
3	$1 + 4 \times 3 + 4 \times 3 = 25$	$8 \times 3 + 1 = 25$ or $4 \times 6 + 1 = 25$
4	$1 + 4 \times 4 + 4 \times 4 = 33$	$8 \times 4 + 1 = 33$ or $4 \times 8 + 1 = 33$

b. Jeff's rule is: the number of tiles on the top face (1) plus the number of tiles on the 4 side faces of each cube multiplied by the number of cubes plus the number of landing platforms multiplied by the number of tiles on each platform (4). If n stands for the number of cubes for the spacecraft, then the number of square tiles in Jeff's rule is $1 + 4 \times n + 4 \times n$ or $1 + 4n + 4n$.

Two possible rules for Fran are: the number of square tiles is 1 more than 8 times the number of cubes, or the number of square tiles is 1 more than 4 times twice the number of cubes. So for any number (n) of stacked cubes, there are $8 \times n + 1$ or $4 \times 2n + 1$ square tiles.

c. For a 100-cube spacecraft, $n = 100$.
So using Jeff's rule:
$1 + 4 \times n + 4 \times n$
$= 1 + 4 \times 100 + 4 \times 100$
$= 1 + 400 + 400$
= 801 (the number of square tiles)

and using Fran's rules:
$8 \times n + 1$
$= 8 \times 100 + 1$
= 801 (the number of square tiles)
or
$4 \times 2n + 1$
$= 4 \times 200 + 1$
$= 801$

d. Fran's first rule indicates that the number of square tiles, 193, is 8 times the number of cubes plus 1. 8 times the number of cubes equals 192, and the number of cubes is 192 ÷ 8 = 24. So the spacecraft is a 24-cube spacecraft.

Page 14 Domino Stacks

ACTIVITY

1. a. 9 dominoes

b. A possible rule is: the number of dominoes in the bottom storey = 2 x number of storeys + 1

c. 41 dominoes. (2 x 20 + 1)

d. 34 storeys. ((69 – 1) ÷ 2)

2. a.

Number of storeys	Number of dominoes
1	3
2	8
3	15
4	24
5	35
6	48
7	63
8	80

b. 120 dominoes

c. i. Answers may vary. One possible rule is: the total number of dominoes equals the number of storeys multiplied by 2 more than the number of storeys.

ii. The following rule is based on the rule above: for n storeys, there will be $n \times (n + 2)$ dominoes.

d. i. 5 928 dominoes. (76 x (76 + 2))

ii. 69 storeys. (A trial-and-improvement strategy is needed here.)

Page 15 Counting Cubes

ACTIVITY

1. a. Vyshan's pattern:
Pattern 1 has 7 cubes.
Pattern 2 has 32 cubes.
Pattern 3 has 81 cubes.

Hema's pattern:
Pattern 1 has 20 cubes.
Pattern 2 has 32 cubes.
Pattern 3 has 44 cubes.

b. Each building has 12 edges surrounding the hollow space. The cubes are increasing by 12 because 1 cube is added to each of the 12 edges to make the next building. A possible rule is: the number of cubes in the previous model plus 12.

Each pattern has 8 corner cubes. The number of cubes that link each corner grows with each pattern and is the same number as the pattern number. So a rule for the number of cubes in each pattern can be expressed as "12 times the pattern number + 8".

2. 216 cubes. (Hema will need 56 cubes, and Vyshan will need 160 cubes (64 + 96).)

3. a. Vyshan's first pattern fits into Hema's first pattern to make a 3 by 3 by 3 solid cube, his second pattern fits into Hema's second pattern to make a 4 by 4 by 4 solid cube, and so on.

b. Hema's next building will have 6 cubes along each edge. Vyshan's next building fits inside Hema's building to make a solid cube, so they will need a total of 6 x 6 x 6 (or 6^3) = 216 cubes.

Page 16 From One to Another

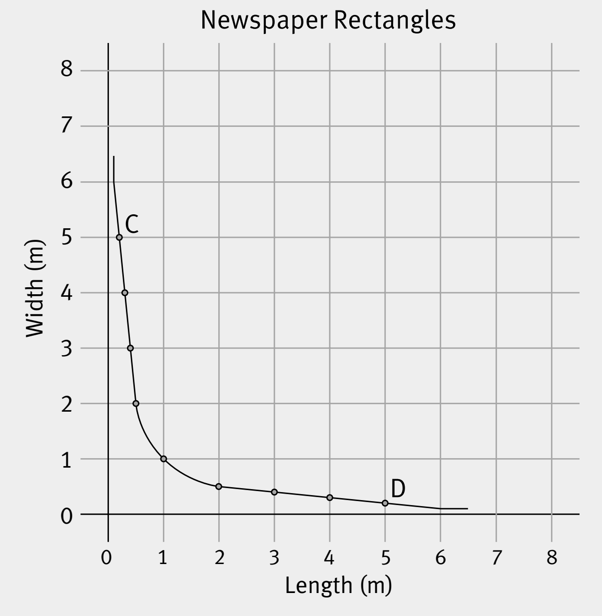
ACTIVITY

1. a. Practical activity

b. Answers may vary. The most likely rectangle will be 2.0 m by 0.5 m.

c. Practical activity

2. a. A possible graph is:



b. Explanations will vary. The points form the path of a hyperbola. (A hyperbola is a curve that gets closer to each axis as the numbers along the axes get larger.) As the length increases, the width decreases (and vice versa). The area remains 1 square metre whatever the rectangle. So the width of a rectangle can never be zero, although it can be almost zero. As the width gets close to zero, the length becomes very large.

c. Two answers are possible. Point C on the graph represents a rectangle with length 20 cm and width 5 m. Point D represents a rectangle with length 5 m and width 20 cm.

Page 17 Areas of Interest

ACTIVITY

1. a.

	Area in square units for:			
	Pattern 1	Pattern 2	Pattern 3	Pattern 4
i.	2	4.5	8	12.5
ii.	2.5	6	10.5	16
iii.	6	10	14	18
iv.	1	3	6	10

b. Answers may vary. Possible rules include:

i. The shaded section is half the area of the surrounding square.

ii. The area of the shaded section is half the area of the surrounding square minus the area of the left-hand corner triangle, which is always 2 units^2.

iii. The area of the shaded section is the area of the surrounding square minus the area of the centre square (the square of the pattern number) minus the area of the 4 corner triangles or half-squares (which is always 2 units^2). The area of the shaded square can also be viewed as 4 times the pattern number plus 2 (the 4 half-squares).

iv. The area of the shaded section is half the area of the surrounding square minus the area of the narrow (unshaded) triangle. The area of this narrow triangle is $\frac{\text{pattern number} + 1}{2}$ units^2.

c. i. 60.5 units^2. The tenth pattern is an 11 by 11 square with an area of 121 units^2. Half of this area is 60.5 units^2.

ii. 70 units^2. The tenth pattern is a 12 by 12 square with an area of 144 units^2. So the area of the shaded section is $^1/_2(144) - 2 = 70$ units^2.

iii. 42 units^2. The tenth pattern is a 12 x 12 square with an area of 144 units^2. It will have a shaded section of 4 x 10 + 2 = 42 units^2.

iv. 55 units^2. The tenth pattern is an 11 x 11 square with an area of 121 units^2. The area of the shaded triangle is equal to the area of one half of a 10 by 11 rectangle, that is $^1/_2$ of 10 x 11 = 55 units^2.

2. Practical activity

Pages 18–19 Mats, Patterns, and Rules

ACTIVITY

1. a. The smaller table mat. (It has 4 + 4 blue circles and 3 + 3 yellow circles, which is the same as (2 x 4) + (2 x 3) circles.)

b. (2 x 5) + (2 x 4) = 18 circles. It has 5 + 5 blue circles and 4 + 4 yellow circles.

c.

Number of circles on shorter side	Evalesi's short cut	Number of circles
3	(2 x 3) + (2 x 2)	10
4	(2 x 4) + (2 x 3)	14
5	(2 x 5) + (2 x 4)	18
12	(2 x 12) + (2 x 11)	46
34	(2 x 34) + (2 x 33)	134
100	(2 x 100) + (2 x 99)	398

d. The x in the rule $2 \times x + 2 \times (x - 1)$ represents the number of circles in the shorter side of any of Evalesi's table mat designs. So x can stand for 3, 4, 5, 12, 34, 100, and, in fact, *any* number.

2. a. i. Terri's design has 2×4 blue circles, 2×4 orange circles, and two purple circles. So there are $2 \times 4 + 2 \times 4 + 2$ circles altogether, which is $4 \times 4 + 2$.

ii.

Number of circles on shorter side	Short cut for Terri's design	Number of circles
3	4 x 2 + 2	10
4	4 x 3 + 2	14
5	4 x 4 + 2	18
12	4 x 11 + 2	46
34	4 x 33 + 2	134
100	4 x 99 + 2	398

iii. When there are any number (x) of circles on the shorter side of a table mat with Terri's design, there are $2 \times (x - 1)$ blue circles, $2 \times (x - 1)$ yellow circles, and 2 purple circles. So there are $2 \times (x - 1) + 2 \times (x - 1) + 2$ circles altogether. This is $4 \times (x - 1) + 2$, or $4(x - 1) + 2$.

b. A possible rule is: $2(x - 2) + 2(x - 1) + 4$. When there are any number (x) of circles on the shorter side of a table mat with Waione's design, there are $2 \times (x - 2)$ yellow circles, $2 \times (x - 1)$ orange circles, and 4 purple circles. So there are $2 \times (x - 2) + 2 \times (x - 1) + 4$ circles altogether.

c. i. The value in cell A2 is 5. So the formula 4*(A2–1)+2 calculates $4 \times (5 - 1) + 2 = 18$. Waione's short cut when $x = 5$ is $2 \times (5 - 2) + 2 \times (5 - 1) + 4$, which can be simplified to $4 \times (5 - 1) + 2$. So the spreadsheet formula is the same as Waione's version of the algebraic rule.

ii. A formula based on the rule given in **b** is: =2*(A2–2)+2*(A2–1)+4

iii. The spreadsheet should look like this:

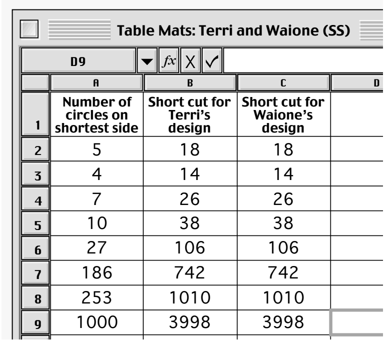


Table Mats: Terri and Waione (SS)

D9

	A	B	C	D
1	Number of circles on shortest side	Short cut for Terri's design	Short cut for Waione's design	
2	5	18	18	
3	4	14	14	
4	7	26	26	
5	10	38	38	
6	27	106	106	
7	186	742	742	
8	253	1010	1010	
9	1000	3998	3998	

3. a. 2. In this rule, x stands for the number of circles on the shorter side of any of the mat designs. So, for example, when $x = 7$, $4 \times x - \square = 26$ becomes $4 \times 7 - \square = 26$. This is $28 - \square = 26$, so the value in the box is 2. Evalesi's rule is therefore $4 \times x - 2$, or $4x - 2$.

b. i. $2 \times x + 2 \times (x - 1) = 2 \times x + 2 \times x - 2 \times 1$
$= 2x + 2x - 2$
$= 4x - 2$

ii. $4 \times (x - 1) + 2 = 4 \times x - 4 \times 1 + 2$
$= 4x - 4 + 2$
$= 4x - 2$

iii. $2 \times (x - 2) + 2 \times (x - 1) + 4$
$= 2x - 4 + 2x - 2 + 4$
$= 2x + 2x - 4 - 2 + 4$
$= 4x - 2$

Pages 20–21 Rotten Apples

ACTIVITY

1. a. 10 weeks

b. Tray A: 5 weeks
Tray B: 6 weeks
Tray C: 7 weeks
Tray D: 8 weeks
Tray E: 9 weeks

c. i. Telea's rule could be: the maximum number of weeks it takes all the apples to become rotten is 1 fewer than the number of apples in the tray.

ii. If a tray holds 15 apples, the longest time it takes for all the apples to become rotten is 14 weeks. So, if a tray holds any number (n) of apples, the longest time it takes for all the apples to become rotten is $n - 1$ weeks.

d. Kathy is correct. For an odd number of apples, placing the rotten apple in the middle will give the minimum amount of time for all the other apples to become rotten because there is the same number of apples on either side of the middle rotten apple. When a tray has n apples altogether, $n - 1$ is the number of apples minus the middle apple and $\frac{n-1}{2}$ stands for the number of apples on either side of the middle apple. For example, if there are 11 apples, there are $(11 - 1) \div 2 = 5$ apples on either side of the middle apple, so they all become rotten in 5 weeks. For n apples, there will be $\frac{n-1}{2}$

apples on either side of the middle apple, and the minimum time they will take to all become rotten is $\frac{n-1}{2}$ weeks.

e. A possible rule is $n \div 2$ or $\frac{n}{2}$ weeks. (With an even number of apples, for example, 20, in the tray, the minimum "rotting time" will occur when the rotten apple is as close to the middle as possible. There is no exact middle, so there are 10 apples on one side and 9 on the other side of the rotten apple. The 10 apples on one side of the rotten apple will take 10 weeks (20 ÷ 2) to become rotten. This is 1 week more than for the 9 apples on the other side.)

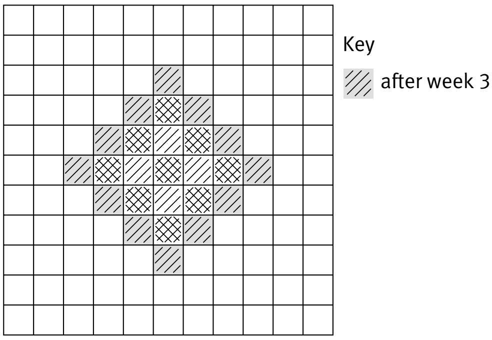
f. i. Minimum: 22; maximum: 43

ii. Minimum: 21; maximum: 41

iii. Minimum: 102; maximum: 203

iv. Minimum: 101; maximum: 201

2. a. i.



There will be 12 new rotten apples (shaded) after week 3.

ii. 41

b. i. 4 x 7 = 28 new rotten apples

ii. The formula =C2+B3 adds the value in cell C2 to the value in cell B3 and puts the answer in cell C3 to give the total number of rotten apples after 1 week.

iii.

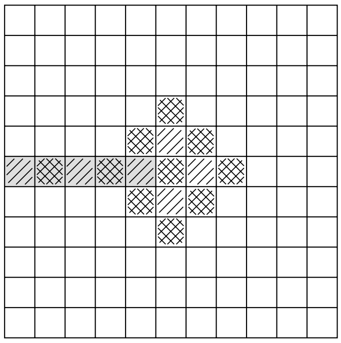
Kathy's Spreadsheet (SS)

C14 =C13+B14

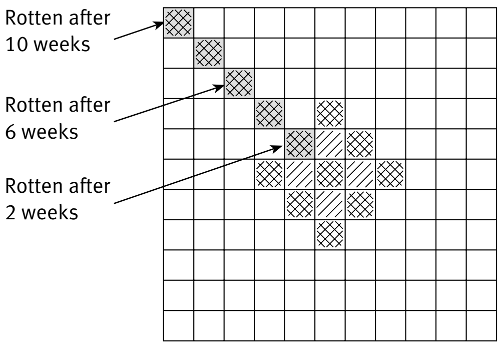
	A	B	C
1	Week	Number of new rotten apples	Total number of rotten apples
2	0	1	1
3	1	4	5
4	2	8	13
5	3	12	25
6	4	16	41
7	5	20	61
8	6	24	85
9	7	28	113
10	8	32	145
11	9	36	181
12	10	40	221
13	11	44	265
14	12	48	313

c. i. Kathy knows that there are 121 apples in the 11 by 11 square tray. Her spreadsheet shows that 113 apples are rotten after 7 weeks and that 145 apples are rotten after 8 weeks. So she thinks that it will take 8 weeks for the total of 121 apples in the tray to become rotten.

ii. Telea notices 5 spaces for apples on either side of the middle rotten apple. It takes 5 weeks for the apples in these spaces to all become rotten. So Telea thinks that all the apples in the tray will be rotten in 5 weeks.



iii. Both Kathy and Telea are incorrect. In the diagram below, it takes 2 weeks for the apple in the shaded square nearest the middle apple to become rotten. It takes a further 2 weeks for the apple in the next shaded space in the diagonal to become rotten. The apple in each additional shaded space in the diagonal takes a further 2 weeks than the apple in the previous shaded space in the diagonal to become rotten. There are 5 such shaded spaces in the tray, so all the apples in the tray will become rotten in 5 x 2 = 10 weeks.



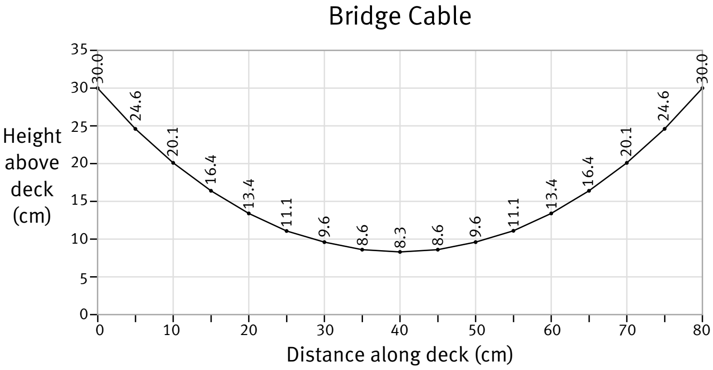
Pages 22–23 Suspended Thought

ACTIVITY

1. **a.–c.** Practical activity. The results come from actual measurements, so they will vary. A possible set of results is shown in the table below.

Distance along bridge (cm)	Depth of sag (cm)	Height of cable above deck (cm)
0	0.0	30.0
5	5.4	24.6
10	9.9	20.1
15	13.6	16.4
20	16.6	13.4
25	18.9	11.1
30	20.4	9.6
35	21.4	8.6
40	21.7	8.3
45	21.4	8.6
50	20.4	9.6
55	18.9	11.1
60	16.6	13.4
65	13.6	16.4
70	9.9	20.1
75	5.4	24.6
80	0.0	30.0

2. The shape of the catenary based on the measurements above is shown below.



3. Once the height of the towers is decided, the length of each vertical suspender cable is the tower height minus the depth of sag at that point. The bridge in this activity has towers that are 30 metres high. In each case, the depth of sag is subtracted from 30, as in the last column of the table in question **1.** An engineer could use a graph like this by converting each y co-ordinate to metres. (The y co-ordinates are the values in the last column of the table.)

4. As the length of the cable increases, so does the depth of sag. Also, the sides of the curve become steeper.

INVESTIGATION

Results will vary.

The sides of a suspended rope or chain hang more steeply than the centre because the links closer to the ends are supporting the greatest mass, so the effect of gravity is felt most strongly by them.

There are many websites about the Golden Gate Bridge. One that includes an animated diagram showing the order in which the parts of the bridge were put together is:
www.orbitforkids.com/yellow/dt_yellow_golden.html

Page 24 Jam Jars

ACTIVITY

1. **i.** b

   ii. c

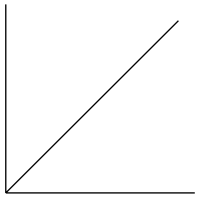
   iii. e

   iv. d

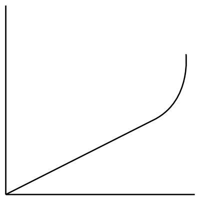
   v. a

2. Your graphs should be similar to the following:

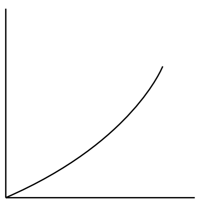
   a.



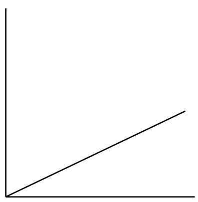
   b.



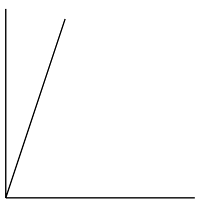
   c.



   d.



   e.



Teachers' Notes

Overview Algebra: Book Four (level 4+)

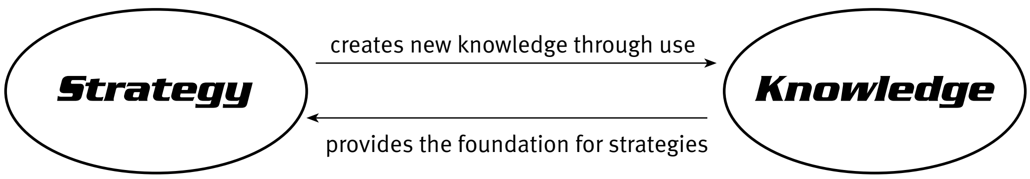
Introduction to Algebra

Teaching and learning algebra has always posed difficulties for teachers and their students. Even today, there is no consensus about when it should be introduced and exactly what should be included in an introduction to algebra. Historically, algebra has formed an important part of the secondary curriculum. However, its inclusion as a strand in the national curriculum statement, *Mathematics in the New Zealand Curriculum,* has meant that teachers at all levels have been grappling with what should be taught at these levels. Internationally, there is a growing consensus that the ideas of algebra have a place at every level of the mathematics curriculum.

One view is that algebra is an extension of arithmetic. Another view is that it is a completion of arithmetic. Some argue that algebra begins when a set of symbols is chosen to stand for an object or situation. Others argue that all basic operations are algebraic in nature: for example, the underlying structure of a part-whole mental strategy for adding 47 to 36 is algebraic because it may involve seeing 47 + 36 as 47 + 33 + 3, giving 50 + 30 + 3 and then 83, and such mental action constitutes algebraic thinking, in spite of the absence of algebraic-looking symbols.

The Figure It Out series aims to reflect the trends in modern mathematics education. So this series promotes the notion of algebraic thinking in which students attend to the underlying structure and relationships in a range of mathematical activities. While the student material includes only limited use of algebraic symbols, the teachers' notes show how mathematical ideas formulated in words by learners can be transformed into symbolic form. Teachers are encouraged to introduce the use of symbols in cases where they themselves feel comfortable and where they think that their students are likely to benefit. It is intended that with this book (Book Four), most students will do some work with algebraic symbols.

The basis for the material in the students' books is consistent with the basis of the Number Framework, which highlights the connections between the strategies students use to explore new situations and the knowledge they acquire.



To help students develop sensible strategies or short cuts for working with new mathematical situations, the activities encourage them to create their own visual and pictorial images to represent mathematical ideas and relationships. These strategies can then be applied to a range of similar, as well as new, mathematical situations.

There are four *Algebra* books in this series for year 7–8 students:
Link (Book One)
Level 4 (Book Two)
Level 4 (Book Three)
Level 4+ (Book Four)

Page 1 Square Number Differences

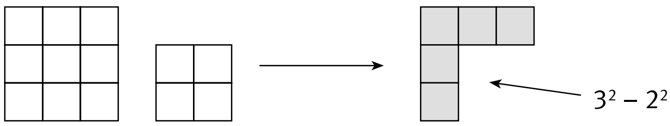
Achievement Objectives

- find a rule to describe any member of a number sequence and express it in words (Algebra, level 4)
- use a rule to make predictions (Algebra, level 4)

ACTIVITY

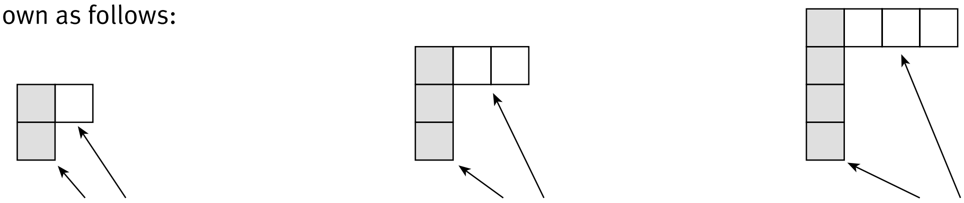
In this activity, students look for patterns in square tile arrangements. Showing differences with tiles, multilink cubes, or drawings on squared paper is likely to help the students see the various patterns.

In question **1**, the tile arrangements relate to the differences between consecutive square numbers. When 3^2, represented by a 3 by 3 square, is compared with 2^2 (a 2 by 2 square), the difference is 5 squares, represented as an L-shaped design.



3^2 2^2 Difference between 3^2 and 2^2 is 5 (5 squares)

The differences between two consecutive square numbers, for example, $2^2 - 1^2$, $3^2 - 2^2$, or $4^2 - 3^2$, can be shown as follows:

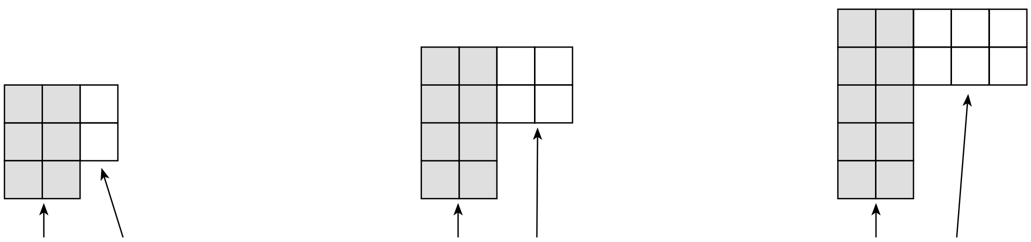


$2^2 - 1^2 = 2 + 1$ $3^2 - 2^2 = 3 + 2$ $4^2 - 3^2 = 4 + 3$

By considering the equations or the diagrams, the students may see a simple rule for these calculations, in which the difference between the numbers being squared is 1. The rule is: add the two numbers that are being squared. For example, $7^2 - 6^2$ can be represented as $7 + 6 = 13$, and $19^2 - 18^2$ as $19 + 18 = 37$. This rule can also be expressed algebraically as: "If $b = a + 1$ (where a and b are consecutive numbers), then $b^2 - a^2 = a + b$." The diagrams above should help the students to understand geometrically why the rule works. For example, $20^2 - 19^2 = 20 + 19$ (which is $b^2 - a^2 = a + b$ for the values $a = 19$, $b = 20$)

$= 39$

In question **2**, the pattern for the differences between alternate square numbers can be shown as follows:



$3^2 - 1^2 = 3 \times 2 + 2 \times 1$
$= 2 \times (3 + 1)$
$= 8$

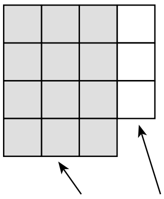
$4^2 - 2^2 = 4 \times 2 + 2 \times 2$
$= 2 \times (4 + 2)$
$= 12$

$5^2 - 3^2 = 5 \times 2 + 2 \times 3$
$= 2 \times (5 + 3)$
$= 16$

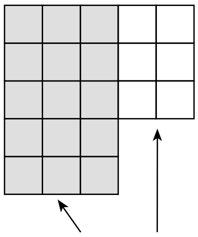
The students may see a simple rule for these calculations, in which the difference between the numbers that are squared is 2. The rule is: double the sum of the two numbers that are squared. So, for example, $27^2 - 25^2 = 2 \times (27 + 25)$.

Once again, the diagrams above will help the students to understand geometrically why the rule works. Algebraically, the rule here is: if $b = a + 2$, then $b^2 - a^2 = 2(a + b)$.

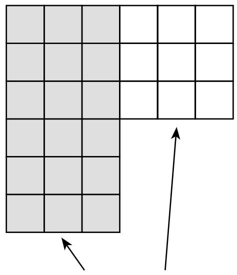
In question **3**, the pattern for the difference between any square number and the third following square number can be shown as follows:



$4^2 - 1^2 = 4 \times 3 + 3 \times 1$
$= 3 \times (4 + 1)$
$= 15$



$5^2 - 2^2 = 5 \times 3 + 3 \times 2$
$= 3 \times (5 + 2)$
$= 21$



$6^2 - 3^2 = 6 \times 3 + 3 \times 3$
$= 3 \times (6 + 3)$
$= 27$

The students may see a simple rule for these calculations, in which the difference between the numbers that are squared is 3. The rule is: treble the sum of the two numbers that are squared. So, for example, $27^2 - 24^2 = 3 \times (27 + 24)$. Algebraically, the rule here is: if $b = a + 3$, then $b^2 - a^2 = 3(a + b)$.

These rules can be further generalised. The following tables summarise the calculations for questions **1** to **3**. For each of these tables, the short cut calculation is always the difference multiplied by the sum of the two numbers that are squared. Note that in the first table below, the difference is 1, so we can write 4 + 3 instead of 1 x (4 + 3), and so on.

Difference of 1:

Difference of squares	Short cut calculation	Pattern in calculation
$3^2 - 2^2$	$3 + 2$	$(3 - 2) \times (3 + 2)$
$4^2 - 3^2$	$4 + 3$	$(4 - 3) \times (4 + 3)$
$5^2 - 4^2$	$5 + 4$	$(5 - 4) \times (5 + 4)$
$10^2 - 9^2$	$10 + 9$	$(10 - 9) \times (10 + 9)$
$b^2 - a^2$	$1 \times (b + a)$	$(b - a) \times (b + a)$

Difference of 2:

Difference of squares	Short cut calculation	Pattern in calculation
$4^2 - 2^2$	$2 \times (4 + 2)$	$(4 - 2) \times (4 + 2)$
$5^2 - 3^2$	$2 \times (5 + 3)$	$(5 - 3) \times (5 + 3)$
$6^2 - 4^2$	$2 \times (6 + 4)$	$(6 - 4) \times (6 + 4)$
$11^2 - 9^2$	$2 \times (11 + 9)$	$(11 - 9) \times (11 + 9)$
$b^2 - a^2$	$2 \times (b + a)$	$(b - a) \times (b + a)$

Difference of 3:

Difference of squares	Short cut calculation	Pattern in calculation
$5^2 - 2^2$	$3 \times (5 + 2)$	$(5 - 2) \times (5 + 2)$
$6^2 - 3^2$	$3 \times (6 + 3)$	$(6 - 3) \times (6 + 3)$
$7^2 - 4^2$	$3 \times (7 + 4)$	$(7 - 4) \times (7 + 4)$
$12^2 - 9^2$	$3 \times (12 + 9)$	$(12 - 9) \times (12 + 9)$
$b^2 - a^2$	$3 \times (b + a)$	$(b - a) \times (b + a)$

In general, then, the rule is: *b* squared minus *a* squared equals the difference between *b* and *a* multiplied by the sum of *b* and *a*. Algebraically, we can write this as:

$b^2 - a^2 = (b - a) \times (b + a)$
$= (b - a)(b + a)$.

This rule will work for all values of *a* and *b*. The following table shows how this rule can be used:

Difference of squares	Calculation
$7^2 - 2^2$	(7 – 2) x (7 + 2) = 45
$10^2 - 3^2$	(10 – 3) x (10 + 3) = 91
$20^2 - 4^2$	(20 – 4) x (20 + 4) = 384
$109^2 - 9^2$	(109 – 9) x (109 + 9) = 11 800

Pages 2–3 Calendars and Short Cuts

Achievement Objectives

- find a rule to describe any member of a number sequence and express it in words (Algebra, level 4)
- use a rule to make predictions (Algebra, level 4)

ACTIVITY

In this activity, students initially investigate short cuts for adding up sets of numbers shown on different calendar months. They then use these short cuts to devise rules for adding sets of numbers on hundreds boards, where the numbers are arranged consecutively from 1 to 100.

In question **1**, the students look for a short cut to add 4 numbers arranged in a square.

1	2
8	9

The numbers in this arrangement were chosen because students usually see quickly that the numbers in the diagonals add up to 10.

When they have completed question **1b**, encourage the students to write a simple rule for adding any set of 4 calendar numbers arranged in a square. One rule is: add the pairs of numbers in a diagonal and double the result.

The following arrangements, based on the difference of 7 (because there are 7 days in a week), help show why this rule works.

1	2
8	9

→

1	1 + 1
1 + 7	1 + 7 + 1

Note that the sum of the numbers in each diagonal above is 1 + 1 + 7 + 1 = 10, giving the total as 2 x 10 = 20. The sum of the numbers in each diagonal below is 15 + 15 + 7 + 1 = 38, giving the total as 2 x 38 = 76.

15	16
22	23

→

15	15 + 1
15 + 7	15 + 7 + 1

In general, any 4 numbers arranged in a square on a calendar grid will have the form:

n	$n + 1$
$n + 7$	$n + 7 + 1$

So each diagonal will add up to $n + n + 7 + 1 = 2n + 8$, and the sum of both diagonals will be $2 \times (2n + 8) = 4n + 16$.

Students who follow this reasoning may use 2 x (2n + 8) = 4n +16 to develop two alternative rules. One rule would be: double the result of adding 8 to twice the first number (2 x (2n + 8)). The second rule would be: add 16 to 4 times the first number (4n + 16).

For example, the first number in this set of 4 numbers is 20:

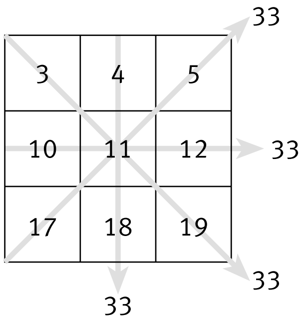
20	21
27	28

So the sum of the 4 numbers is:
2 x (2 x 20 + 8) = 2 x 48
= 96
or 4 x 20 + 16 = 96

The arrangements below help to explain the short cut 9 x 11 used by Anna to add the 9 numbers shown in the calendar for question **2**.

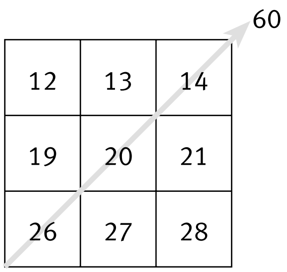
3	4	5
10	11	12
17	18	19

The students may notice that the numbers in each diagonal add up to 33, which is 3 x 11, and that the numbers in the centre column and the numbers in the centre row also each add up to 33. The question, then, is how to make use of this information. One approach is to add the results for the two diagonals to the sums for the centre column and for the centre row.



The students will see that this approach counts every cell in the grid once, except for the middle cell, which gets counted 4 times. (A cell is counted once for every line that passes through it.) So the total of all cells is: 33 + 33 + 33 + 33 – 3 x 11 (the sum of the 2 diagonals, the middle row, and the middle column minus 3 times the middle cell), which is 4 x 33 – 1 x 33, or 3 x 33 = 99. So a rule might be: 3 times the sum of the diagonal. We could also use the middle column or middle row.

The arrangement below shows how the short cuts work for question **2c iii**.



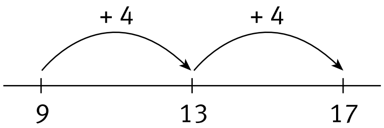
20 – 8	20 – 7	20 – 6
20 – 1	20	20 + 1
20 + 6	20 + 7	20 + 8

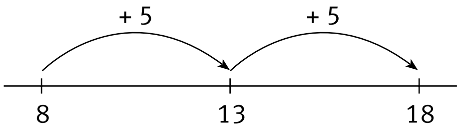
The total of the numbers is 3 x 60 = 180 or, using Anna's short cut, 9 x 20 = 180. Note that in each of the examples for question **2**, short cuts lead to a rule for any similar arrangement of 9 numbers. The simplest rule, using Anna's short cut, is: multiply the middle number by 9.

In question **3**, Angus uses $\frac{1 + 25}{2}$ x 16 to calculate the total of the 16 numbers in the arrangement below. Anna used the rule “multiply the middle number by 9” for calendar arrangements with 9 numbers. So Angus extends the rule to “multiply the middle number by 16” for calendar arrangements with 16 numbers.

1	2	3	4
8	9	10	11
15	16	17	18
22	23	24	25

Note that, with 16 numbers, the middle number (in this case, 13) does not show within the grid. The students may need to convince themselves that 13 is indeed the middle number for any “balanced” pair of cells in this grid. For example, for 9 and 17 or 8 and 18:





Two ways to calculate the middle number are shown in the Answers. The second arrangement below confirms that the total, 208, is 16 lots of 13.

1	2	3	4
8	9	10	11
15	16	17	18
22	23	24	25

13

13 – 12	13 – 11	13 – 10	13 – 9
13 – 5	13 – 4	13 – 3	13 – 2
13 + 2	13 + 3	13 + 4	13 + 5
13 + 9	13 + 10	13 + 11	13 + 12

13

The rule can also be applied to the set of 4 numbers in question **1**. The middle number is hidden but can be calculated by working out the mean (average) of either 1 and 9 or 2 and 8. So the total of the 4 numbers is 4 x 5 = 20 (the middle number multiplied by 4).

1	2
8	9

1	2
8	9

5

In question **3c**, the students use the following rule to calculate the totals:

$\frac{\text{first number + last number}}{2}$ x number of numbers.

So, for example, in question **3c iii**, there are 31 numbers to be added. The first number is 1, and the last is 31. So the total is $\frac{1 + 31}{2}$ x 31 = 496.

In question **4**, the calendar short cuts and the rule that applies are extended to arrays of numbers that are located on a hundreds board. Short cuts such as those shown in the Answers lead to the algebraic rule for calculating the sum of consecutive whole numbers from 1 to any value, *n*.

The rule is: $\frac{1 + n}{2} \times n = n(\frac{n + 1}{2})$.

So, for example, if n = 100, the sum 1 + 2 + … + 99 + 100 can be calculated as

$$100 \times \frac{(100 + 1)}{2} = 100 \times 50.5 = 5\,050$$

Pages 4–5 **Number Juggling**

Achievement Objectives

- devise and use problem-solving strategies to explore situations mathematically (Mathematical Processes, problem solving, level 4)
- record information in ways that are helpful for drawing conclusions and making generalisations (Mathematical Processes, communicating mathematical ideas, level 4)
- interpret information and results in context (Mathematical Processes, developing logic and reasoning, level 4)

ACTIVITY

In this activity, students follow sets of instructions to verify two mathematical claims. In Challenge One, it is claimed that the answer is always 4, no matter what starting number is used, and in Challenge Two, it is claimed that the answer is always 9. (If the students try this with 1 or 2 as the starting number, they will find that this is also true when negative numbers are involved.)

The challenges presented here rely on generating an answer that does not depend on the starting number chosen. In each case, the process will first disguise the chosen number and then "strip away" this initial number to leave the final answer.

For example, a simple challenge would be:
Step 1: think of a number
Step 2: multiply by 6
Step 3: divide your result by twice the number you first thought of.
Your answer must be 3.

If the number chosen is 4, then we have:
Step 1: 4
Step 2: $4 \times 6 = 24$
Step 3: $24 \div (2 \times 4) = 3$

In this case, the original number is disguised in some way (it is multiplied by 6 in step 2) and then the original number is removed from the result (by division in step 3).

Algebraically, if n is our chosen number, we have $6n$ at the end of step 2 and $6n \div 2n = 3$ at the end of step 3.

The example shown above is simpler than those in challenges One and Two, but the principle is the same.

In Challenge One, students might set out a table as shown in the Answers to check the results for several different starting numbers. A further row, which shows the general rule, is:

Step 1	Step 2	Step 3	Step 4
Starting number	Multiply by 2	Square	Divide by square of starting number
n	$n \times 2 = 2n$	$(2n)^2 = 4n^2$	$4n^2 \div n^2 = 4$

The advantage of considering the situation algebraically is that it allows us to see clearly how the process used in the challenge works.

For example, both steps 2 and 3 disguise the initial number (first by multiplying by 2 and then by squaring the result) so that by the end of step 3, we have a number equal to $4n^2$. In order to remove the initial number from the result, we divide by n^2 to give an answer of 4.

Another useful tool to show how the challenge works is a flow chart. The flow chart below shows how the sequence of instructions in Challenge One works.

$$\begin{tikzcd} n \arrow[r, bend left, "\text{x } 2"] & 2n \arrow[r, bend left, "(\ )^2"] & 4n^2 \arrow[r, bend left, "\div n^2"] & 4 \end{tikzcd}$$

The students might follow the flow chart steps, choosing any value for the starting number, *n*. The next flow chart shows the steps when the starting number is 9:

$$\begin{tikzcd} 9 \arrow[r, bend left, "\text{x } 2"] & 18 \arrow[r, bend left, "18^2"] & 324 \arrow[r, bend left, "\div 81"] & 4 \end{tikzcd}$$

After they have completed a few flow charts with particular values for *n*, the students could again consider the first flow chart with the aim of reinforcing their understanding of why the challenge works. They will see that an expression involving *n* occurs in each of the first three steps: in step one, *n* is chosen, in steps two and three, *n* is somehow disguised. In step four, *n* is removed from the expression (the expression $4n^2$ is divided by n^2 to give 4) to give the final result.

The table in the Answers helps to show how Challenge Two works. Notice that the smallest starting number is 3. If it were less than 3, step two would lead to a negative value. This would increase the level of difficulty, but it would not alter the mathematics involved.

The results in the table given in the Answers confirm that, for a range of starting numbers, the result is always 9. Although the algebraic "proof" for this result is likely to be beyond most students at this stage, it is nevertheless shown below:

Step	Working and result
1	n
2	$n - 3$
3	$(n - 3)^2 = n^2 - 3n - 3n + 9$ $= n^2 - 6n + 9$
4	$n^2 - 6n + 9 + 6n = n^2 + 9$
5	$n^2 + 9 - n^2 = 9$

The flow chart below shows how the sequence of instructions in Challenge Two works:

$$\begin{tikzcd} n \arrow[r, bend left, "-3"] & n - 3 \arrow[r, bend left, "(\ )^2"] & n^2 - 6n + 9 \arrow[r, bend left, "+6n"] & n^2 + 9 \arrow[r, bend left, "-n^2"] & 9 \end{tikzcd}$$

This next flow chart shows the steps when the starting number is 9.

$$\begin{tikzcd} 9 \arrow[r, bend left, "-3"] & 6 \arrow[r, bend left, "6^2"] & 36 \arrow[r, bend left, "+54"] & 90 \arrow[r, bend left, "-81"] & 9 \end{tikzcd}$$

In the **Investigation** that follows the challenges, the students try to make up number challenges of their own. Flow charts can help the students see the effect of each operation clearly.

Pages 6–7 Initials Logo

Achievement Objectives

- find and justify a word formula which represents a given practical situation (Algebra, level 4)
- find a rule to describe any member of a number sequence and express it in words (Algebra, level 4)
- use a rule to make predictions (Algebra, level 4)

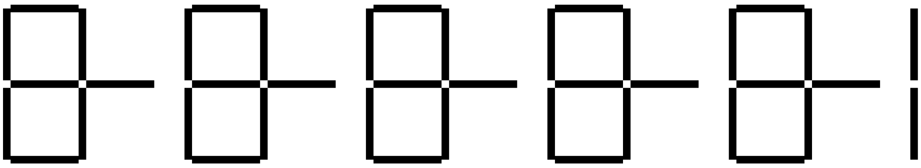
ACTIVITY

In this activity, students use stick logo designs to find short-cut ways to work out the number of sticks needed for any number of joined logos.

In question **1**, Evalesi Henry makes her logo by joining her initials, E and H, and then makes a design by repeating the initials five times.

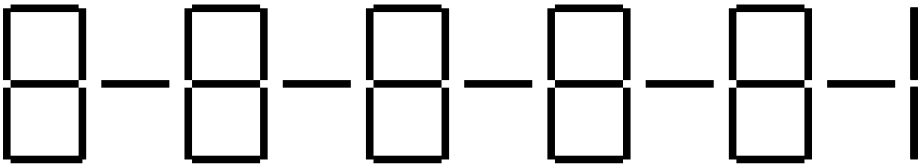
There are several short-cut ways to count the number of sticks in the design above. Three ways, including the one shown in the Answers, are:

Short cut 1



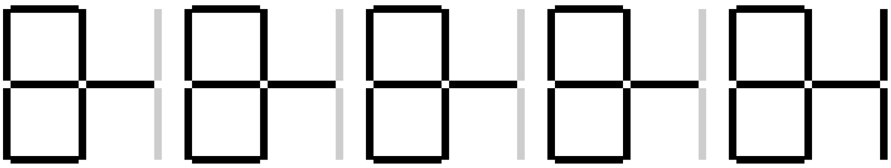
5 x 8 + 2 = 42 sticks

Short cut 2



5 x 7 + 5 + 2 = 42 sticks

Short cut 3



5 x 10 – 4 x 2 = 42 sticks

Notice that, in short cut 3, an additional 4 pairs of sticks (4 x 2 sticks) have been used to complete separate EH logos. They must be subtracted in the short cut because they are not part of the logo design. Each short cut above leads to a rule that can be applied to any design that repeats Evalesi's EH logo in this way.

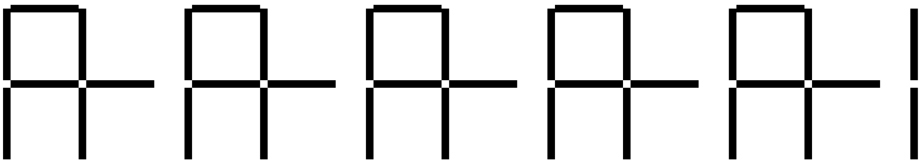
Number of EH logos	Short cut 1	Short cut 2	Short cut 3
1	1 x 8 + 2 = 10	1 x 7 + 1 + 2 = 10	1 x 10 – 0 x 2 = 10
2	2 x 8 + 2 = 18	2 x 7 + 2 + 2 = 18	2 x 10 – 1 x 2 = 18
3	3 x 8 + 2 = 26	3 x 7 + 3 + 2 = 26	3 x 10 – 2 x 2 = 26
4	4 x 8 + 2 = 34	4 x 7 + 4 + 2 = 34	4 x 10 – 3 x 2 = 34
5	5 x 8 + 2 = 42	5 x 7 + 5 + 2 = 42	5 x 10 – 4 x 2 = 42
n	$n \times 8 + 2$ $= 8n + 2$	$n \times 7 + n + 2$ $= 7n + n + 2$ $= 8n + 2$	$n \times 10 - (n - 1) \times 2$ $= 10n - 2(n - 1)$ $= 10n - 2n + 2$ $= 8n + 2$

Although each rule is expressed initially in a different way, they all produce the same result for any given value of n. Therefore, the rules are equivalent. Note how the simplifying process produces $8n + 2$ for each rule.

In questions **2** and **3**, the AH logo produces similar results to those of the EH logo in question **1**. In fact, by noticing that the AH logo has 1 stick fewer than the EH logo, any of the rules for Evalesi's design can be adapted to apply to Arnon's design.

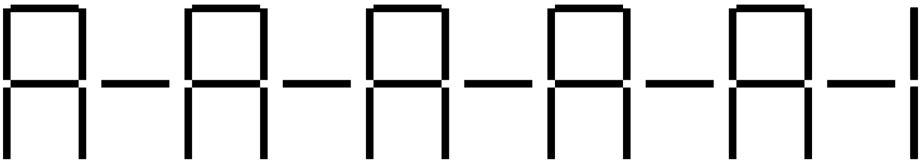
Three short cuts to count the number of sticks in Arnon's design, including the one shown in the Answers, are:

Short cut 4



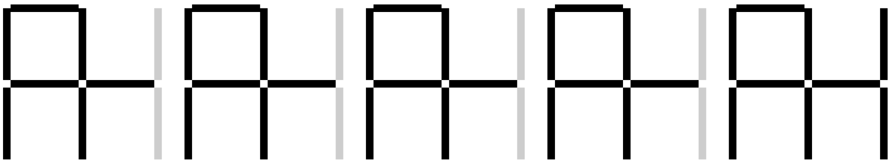
$5 \times 7 + 2 = 37$ sticks

Short cut 5



$5 \times 6 + 5 + 2 = 37$ sticks

Short cut 6



$5 \times 9 - 4 \times 2 = 37$ sticks

Note that, as with short cut 3 for Evalesi's designs, short cut 6 has an additional 4 pairs of sticks (4 x 2 sticks) that have been used to complete separate AH logos. They must therefore be subtracted.

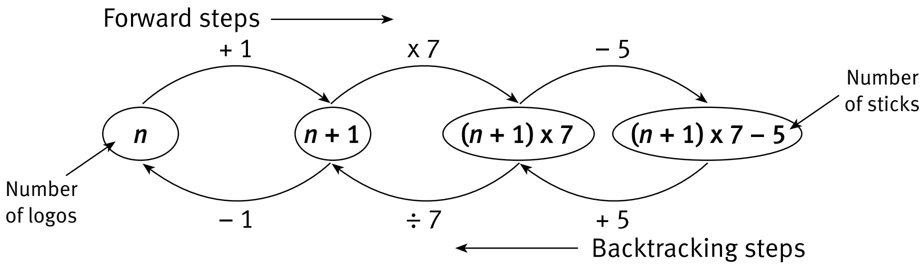
Number of AH logos	Short cut 4	Short cut 5	Short cut 6
1	$1 \times 7 + 2 = 9$	$1 \times 6 + 1 + 2 = 9$	$1 \times 9 - 0 \times 2 = 10$
2	$2 \times 7 + 2 = 16$	$2 \times 6 + 2 + 2 = 16$	$2 \times 9 - 1 \times 2 = 16$
3	$3 \times 7 + 2 = 23$	$3 \times 6 + 3 + 2 = 23$	$3 \times 9 - 2 \times 2 = 23$
4	$4 \times 7 + 2 = 30$	$4 \times 6 + 4 + 2 = 30$	$4 \times 9 - 3 \times 2 = 30$
5	$5 \times 7 + 2 = 37$	$5 \times 6 + 5 + 2 = 37$	$5 \times 9 - 4 \times 2 = 37$
n	$n \times 7 + 2$ $= 7n + 2$	$n \times 6 + n + 2$ $= 6n + n + 2$ $= 7n + 2$	$n \times 9 - (n - 1) \times 2$ $= 9n - 2(n - 1)$ $= 7n + 2$

While some students may need a systematic tabulated approach before they can write these rules algebraically, others may realise very quickly that Arnon's logo has 1 stick fewer than Evalesi's and that, consequently, Evalesi's rule, $8n + 2$, becomes $7n + 2$ for Arnon's design. (As in Evalesi's case, each of Arnon's rules can be simplified to the same rule, as shown above.)

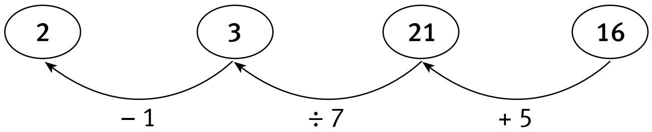
Students who are able to work with this level of algebra may want to design initial logos of their own. They may even like to try working backwards by making logo designs from algebraic rules such as $4n + 3$, $6n + 1$, and so on.

Question **3d** requires the students to use Evalesi's short cut, (n + 1) x 7 – 5, to work backwards from the number of sticks to the number of logos.

In general, for n logos, a forwards-and-backwards flow chart will look like this:



So, for example, when the number of sticks is 16, we have:



So 16 sticks give 2 logos.

The short cuts above lead initially to different rules that can all be applied to any design that repeats Arnon's AH logo in this way.

Pages 8–9 Bathroom Tiles

Achievement Objectives

- find and justify a word formula which represents a given practical situation (Algebra, level 4)
- find a rule to describe any member of a number sequence and express it in words (Algebra, level 4)
- use a rule to make predictions (Algebra, level 4)

ACTIVITY

In this activity, students explore short cuts to work out the number of coloured tiles for different tile designs.

In question **1**, the students are required to analyse how Lis's pattern behaves for different numbers of tiles. (Note that she always starts with pink tiles.) The pattern can be thought of as comprising sets of pink and orange tiles (in groups of 3), together with an additional tile. This last tile is always different in colour to the last set of 3 tiles.

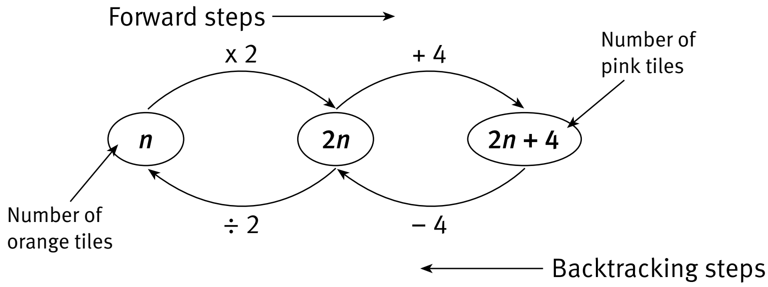
When there are the same number of sets of pink and orange tiles, the equation for the total number of tiles is: total = 3 x number of pink sets + 3 x number of pink sets + 1 pink tile. This number will always be odd and is of the form 6n + 1 (where n is the number of sets of 3 pink tiles).

When there is 1 set fewer of orange tiles than there are sets of pink tiles, the equation is:
total = 3 x number of pink sets + [for the orange sets] 3 x (number of pink sets – 1) + 1 pink tile.
This number will always be even and of the form $3n + 3(n - 1) + 1 = 3n + 3n - 3 + 1$
$= 6n - 2$
(where n is the number of sets of 3 pink tiles).

In question **1c i**, there are 20 sets of 3 tiles, so there are 3 x 10 pink tiles and 3 x 10 orange tiles and the last tile is pink. (Note that the total number of tiles is odd. Using the equation above, we get 6 x 10 + 1 = 61 for the total number of tiles.)

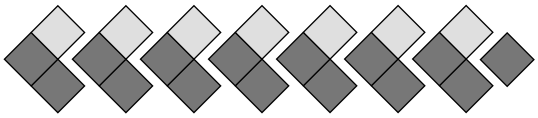
In question **1c ii**, there are 37 sets of 3 tiles, so there are 3 x 19 pink tiles and 3 x 18 orange tiles, and the last tile is orange. (Note that the total number of tiles is even. Using the equation above, we get 6 x 19 – 2 = 112 for the total number of tiles.)

In question **2**, there are 2 orange tiles for each set of 4 pink tiles plus an extra block of 4 pink tiles at the end. The orange tiles are always arranged in pairs. The following "backtracking" flow chart is a useful device for determining the number of pink tiles given the number of orange tiles and vice versa. The number of orange tiles is any even number, *n*. There are twice as many pink tiles as orange tiles, plus a group of 4 pink tiles at the end, so the number of pink tiles is $2n + 4$.



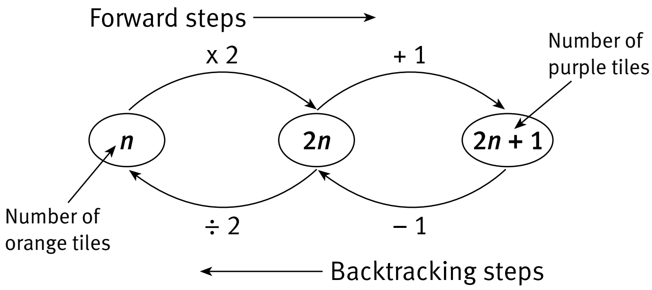
The students could use the flow chart in questions **2b** and **2c** to find the corresponding number of pink or orange tiles and then calculate the total number of tiles.

In Bill's tile designs for question **3**, each orange tile can be linked to 2 purple tiles. There is also 1 additional purple tile.



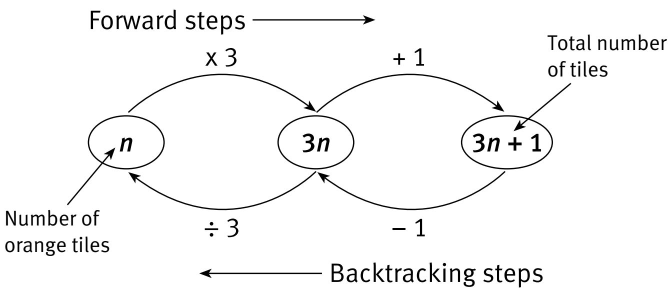
So for a tile section with 100 orange tiles, there are 100 x 2 + 1 purple tiles. For any number of orange tiles, *n*, there will be $n \times 2 + 1 = 2n + 1$ purple tiles. The total number of tiles is therefore *n* orange tiles plus $2n + 1$ purple tiles. This is $n + (2n + 1) = 3n + 1$ tiles altogether.

Some of this work requires "reverse" thinking. For example, in question **3d**, the students are required to calculate how many orange tiles the pattern has if there are 43 purple tiles. In order to do this, they must reverse the order of calculations used in getting from orange tiles to purple tiles. Instead of multiplying by 2 and adding 1, we subtract 1 and divide by 2. The calculation is then $(43 – 1) \div 2 = 21$. The students might find it helpful to devise a backtracking flow chart to generalise the reverse calculations:



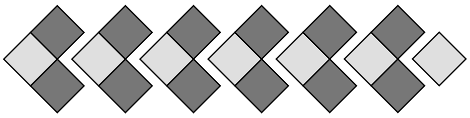
For question **3e**, the students will find it helpful to think of the orange and purple tiles as sets of 3. (They may realise this from their table for **3d**.) So each orange tile x 3 gives a set of 3 (1 orange and 2 purple). There is always an extra purple tile, so the total number of tiles is the number of orange tiles x 3 + 1. Conversely, the total number of tiles – 1 ÷ 3 gives the number of orange tiles.

The backtracking flow chart below will help the students to complete the first table for **3e**:

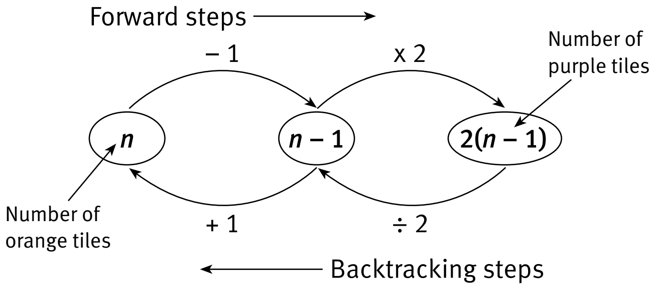


The second table in question **3e** may challenge even able students. There are several approaches to this question, but the reasoning should be along these lines: There is 1 orange tile for each pair of purple tiles and then 1 extra purple tile, so the number of orange tiles equals $^1/_2$ x (the number of purple tiles – 1). One way to find the total number of tiles is to add these quantities. If n is the number of purple tiles, the number of orange tiles is $^1/_2(n - 1)$ and the total number of tiles is $n + {}^1/_2(n - 1)$. As shown in the Answers, another possible rule, where n is the number of purple tiles, is: $(3n - 1) \div 2$. The students should check that this result works (for instance, by using the results of the table in question **3d** or by looking at Bill's designs on the student page).

In question **4**, each of the orange tiles except the final one is linked to a pair of purple tiles. So when there are 7 orange tiles, there are 7 – 1 = 6 pairs of purple tiles.



When there are 100 orange tiles, there are 100 – 1 = 99 pairs of purple tiles. And when the number of orange tiles is any number, n, there will be $n - 1$ pairs of purple tiles. So for n orange tiles, there will be 2 x $(n - 1) = 2(n - 1)$ purple tiles. The backtracking flow chart that could be used to help the students with the reverse calculations in question **4c** is shown below.



The students might like to see if they can find an algebraic rule that will link the total number of tiles to the orange tiles. Such a rule is: if n is the number of orange tiles, the number of purple tiles is $2(n - 1)$, so the total number of tiles is:

$$\begin{aligned} n + 2(n - 1) &= n + 2n - 2 \\ &= 3n - 2. \end{aligned}$$

The students might then like to check their results for the table in question **4c**.

Pages 10–11 Patterns and Designs

Achievement Objectives

- find and justify a word formula which represents a given practical situation (Algebra, level 4)
- find a rule to describe any member of a number sequence and express it in words (Algebra, level 4)
- use a rule to make predictions (Algebra, level 4)

ACTIVITY

In this activity, students devise and use rules for studying patterns in different designs.

In question **1**, the students draw diagrams to show how two different short cuts work (see the Answers). The short cut 2 x 2 + 3 leads to a rule for the total number of coloured squares: double the number of green squares and add 1 more than the number of green squares. This means that if there are x green squares, there are x x 2 + $(x + 1)$ or $2x + x + 1$ coloured squares altogether. So a section of Greer's design that has 73 green squares has a total of 73 x 2 + (73 + 1) = 220 coloured squares.

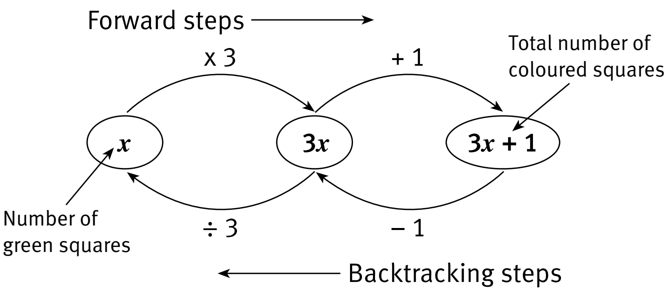
The short cut 2 x 3 + 1 leads to the rule $3x + 1$ for the number of coloured squares when there are x green squares. The first rule, $2x + (x + 1)$, can be simplified as follows: $2x + (x + 1) = 2x + x + 1$

$$= 3x + 1$$

So, although the two rules are expressed differently, they produce the same results for designs with particular numbers of green squares.

In question **1d**, the students will be able to use either of the two rules from the short cuts in questions **1b** and **1c** to find the total number of coloured squares in the first table. The second table requires the students to calculate the number of green squares, given the total number of coloured squares. The students who attempt this using the first rule will find that they are having to guess and check the results. Those who use the second rule, $3x + 1$, will find that the calculations are much more straightforward.

To reverse this rule, the students must first subtract 1 from the value for the total number of coloured squares and then divide the result by 3. The following backtracking flow chart shows how this works:



In question **2b**, the students try to devise their own short cut for poutama designs. Two of the possible short cuts are shown in the Answers. In question **2c**, the students examine Hine's short cut.

The following table shows how the short cuts lead to algebraic rules that all simplify to $5x + 1$.

Number of steps in poutama	Short cut 1	Short cut 2	Short cut 3 (Hine's)
3	3 x 5 + 1 = 16	6 + (2 x 5) = 16	3 x 6 − 2 = 16
4	4 x 5 + 1 = 21	6 + (3 x 5) = 21	4 x 6 − 3 = 21
5	5 x 5 + 1 = 26	6 + (4 x 5) = 26	5 x 6 − 4 = 26
10	10 x 5 + 1 = 51	6 + (9 x 5) = 51	10 x 6 − 9 = 51
x	x x 5 + 1 = $5x + 1$	$6 + (x - 1)$ x 5 $= 6 + 5(x - 1)$ $= 1 + 5x$ $= 5x + 1$	x x 6 − $(x - 1)$ $= 6x - (x - 1)$ $= 5x + 1$

As with the questions above, there are several ways to visualise short cuts and rules for question **3**. Some are shown in the Answers. A further short cut, using additional blocks that must be subtracted, is used by Tracey in question **4**.

The following table shows how the three short cuts used for questions **3** and **4** lead to algebraic rules.

Number of crosses	Short cut 1	Short cut 2	Short cut 3 (Tracey's)
3	16 + 2 x 13 = 42	3 x 13 + 3 = 42	3 x 16 − 2 x 3 = 42
4	16 + 3 x 13 = 55	4 x 13 + 3 = 55	4 x 16 − 3 x 3 = 55
5	16 + 4 x 13 = 68	5 x 13 + 3 = 68	5 x 16 − 4 x 3 = 68
10	16 + 9 x 13 = 133	10 x 13 + 3 = 133	10 x 16 − 9 x 3 = 133
x	$16 + (x - 1)$ x 13 $= 16 + 13(x - 1)$ $= 16 + 13x - 13$ $= 13x + 3$	x x 13 + 3 $= 13x + 3$	x x 16 − $(x - 1)$ x 3 $= 16x - 3(x - 1)$ $= 16x - 3x + 3$ $= 13x + 3$

Alien Critters

Achievement Objectives

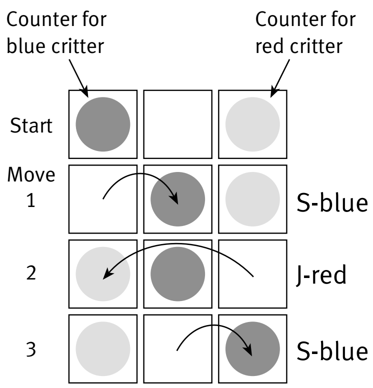
- devise and use problem-solving strategies to explore situations mathematically (Mathematical Processes, problem solving, level 4)
- find a rule to describe any member of a number sequence and express it in words (Algebra, level 4)
- use a rule to make predictions (Algebra, level 4)

ACTIVITY

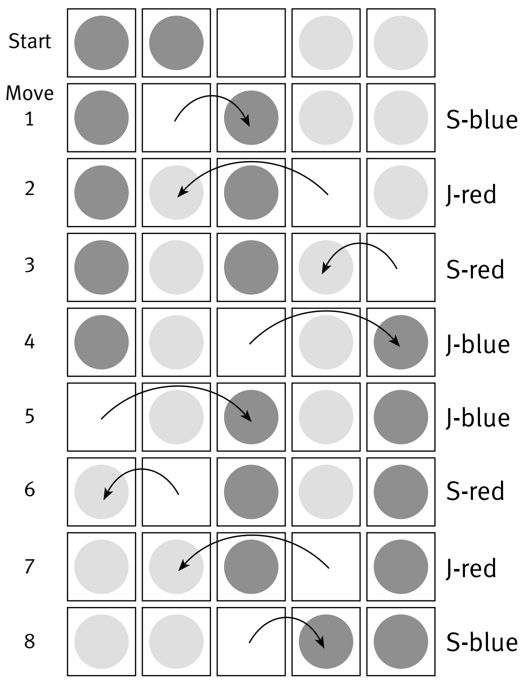
In this activity, students must first systematically explore the moves involving steps, S, and jumps, J, for equal numbers of blue and red alien critters to swap positions on an "asteroid bridge". The bridge always has 1 more asteroid than there are alien critters. So for 3 blue and 3 red alien critters, there are 7 asteroids.

The students will need to use equipment such as pegs on a pegboard or counters on squared paper to actually carry out the moves. They will also need to devise a sensible way to monitor the sequence of moves they make. In the following diagram, S-blue indicates that a step was made by a blue critter, and J-red indicates that a jump was made by a red critter.

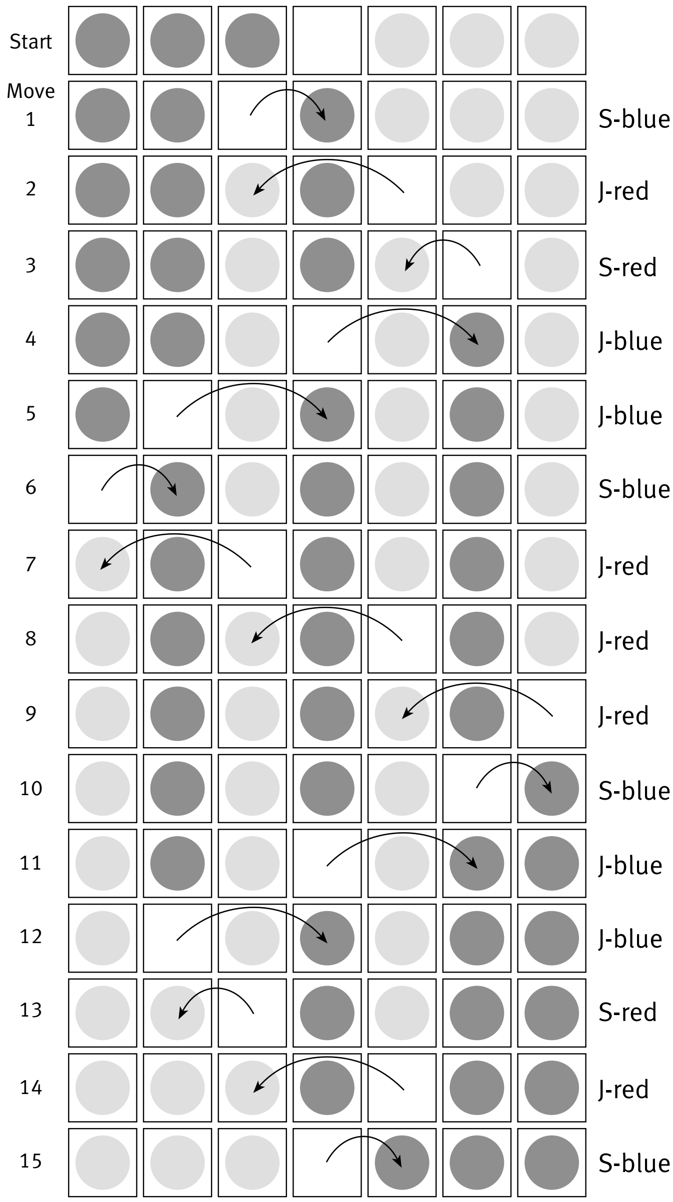
1 blue critter, 1 red critter, 3 asteroids



2 blue critters, 2 red critters, 5 asteroids



3 blue critters, 3 red critters, 7 asteroids



Note that, in each table, there is rotational symmetry about the middle of the table. If each table, excluding any labels and arrows, were rotated a half-turn (180 degrees) about its middle, the table would look exactly as it did prior to the half-turn being made. This kind of symmetry can often be shown when such position swapping occurs.

As indicated, the students will need to work carefully to ensure that the number of steps and jumps they make is accurately determined. It is also important that the students accurately assess the *minimum* number of moves of each type. Combinations of moves that use more than the minimum are also possible, but if the numbers in the table are greater than they ought to be, it is very unlikely that the students will be able to devise rules for the number of steps and jumps. For this reason, it may be helpful to have the students share their results for the minimum number of moves for different numbers of critters.

The completed table for question **2** in the Answers shows that "the minimum number of steps is twice the number of blue or red critters" and that "the minimum number of jumps is the square of the number of blue or red critters". In looking for patterns in the table, the students may come up with alternative rules for the minimum number of steps. For example, "the minimum number of steps is equal to the total number of blue *and* red critters," or "the minimum number of steps is equal to 1 less than the number of asteroids".

They are likely to find the rule for the minimum number of jumps challenging. Encourage them to look for a direct relationship between the number of blue (or red) critters and the minimum number of jumps. Here are these rules shown algebraically:

Number of critters		Number of asteroids	Minimum number of steps	Minimum number of jumps	Minimum total number of moves
Blue	Red				
n	n	$2n + 1$	$2n$	n^2	$n^2 + 2n$

The table for question **3** in the Answers shows an alternative way to generalise the minimum number of moves for any number of blue or red critters. Here is the rule shown algebraically:

Number of blue or red critters	Minimum number of moves (pattern)
n	$n \times (n + 2)$ $= n(n + 2)$

The pattern in the table indicates that the minimum number of moves can be found by multiplying the number of blue (or red) critters by 2 more than the number of blue (or red) critters. So, for example, when there are 23 of each kind of alien critter, the minimum number of moves is 23 x 25 = 575. Note also that $n(n + 2) = n^2 + 2n$. This shows that the two rules are equivalent. So when $n = 23$, the least number of moves can be calculated from either 23 x 25 = 575 or 23 x 23 + 2 x 23 = 575. The students may notice other patterns, for example, that the total number of moves is 1 less than the square of (the number of blue (or red) critters plus 1). Algebraically, the number of moves = $(n + 1)^2 - 1$. So, for example, when $n = 23$, the total number of moves is

$$(23 + 1)^2 - 1 = 24 \times 24 - 1$$
$$= 575.$$

Once again, it can be shown that the rules $(n + 1)^2 - 1$ and $n^2 + 2n$ are equivalent:

$$\begin{aligned}(n + 1)^2 - 1 &= n^2 + n + n + 1 - 1 \\ &= n^2 + 2n + 1 - 1 \\ &= n^2 + 2n\end{aligned}$$

A similar activity involving frogs can be found at www.nzmaths.co.nz/brightsparks

Page 13 Tiling Spacecraft

Achievement Objectives

- devise and use problem-solving strategies to explore situations mathematically (Mathematical Processes, problem solving, level 4)
- find a rule to describe any member of a number sequence and express it in words (Algebra, level 4)
- use a rule to make predictions (Algebra, level 4)
- find and justify a word formula which represents a given practical situation (Algebra, level 4)

ACTIVITY

In this activity, students use different approaches to finding a rule for a practical situation.

In question **1a**, the students must interpret 1 + 4 x 3 + 2 x 3 as being the number of tiles for a 3-cube spacecraft. There is 1 tile for the top face, 4 x 3 tiles for the side faces of the 3 cubes, and 2 x 3 tiles for the 2 landing platforms. (See the diagram in the Answers.) They draw this spacecraft on isometric dot paper. This task is likely to focus their attention on the three different tile positions: the top, sides, and landing platforms. This should help them see how Jeff's rule arises.

The following table shows how Jeff's short cut leads to his rule:

Number of cubes for spacecraft	Number of tiles for: top face	side faces	platforms	spacecraft
1	1	4 x 1	2 x 1	1 + 4 x 1 + 2 x 1 = 7
2	1	4 x 2	2 x 2	1 + 4 x 2 + 2 x 2 = 13
3	1	4 x 3	2 x 3	1 + 4 x 3 + 2 x 3 = 19
4	1	4 x 4	2 x 4	1 + 4 x 4 + 2 x 4 = 25
5	1	4 x 5	2 x 5	1 + 4 x 5 + 2 x 5 = 31
n	1	4 x n = 4n	2 x n = 2n	1 + 4 x n + 2 x n = 1 + 4n + 2n

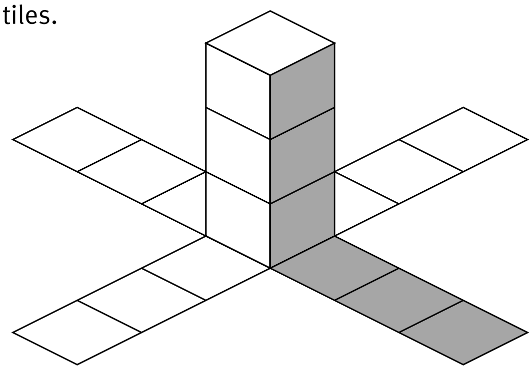
Fran uses an alternative approach in question **3**. She doesn't separate tiles for the top face from the tiles for the side faces and the tiles for the landing platforms as Jeff has. She gets the total number of tiles for spacecraft of different sizes and then looks for a pattern. Her rule, 6 x n + 1 or 6n + 1, is probably based on the fact that the number of tiles increases by 6 for each cube added.

Craft size (number of cubes)	Number of square tiles	Pattern in number of square tiles
1	7	6 x 1 + 1
2	13	6 x 2 + 1
3	19	6 x 3 + 1
4	25	6 x 4 + 1
5	31	6 x 5 + 1
n		6 x n + 1 = 6n + 1

Note that, although the rules are different, Jeff's rule simplifies to 6n + 1, which is the same as Fran's rule.

In question **4**, the rules change in that there are 4 landing platforms instead of 2 as in questions **1** to **3**. Jeff adjusts his rule to allow for the 2 extra landing platforms, while Fran may again look for a direct relationship between the number of cubes and the number of tiles.

Alternatively, Fran might recognise that we can consider each vertical face of the spacecraft together with its associated landing platform.

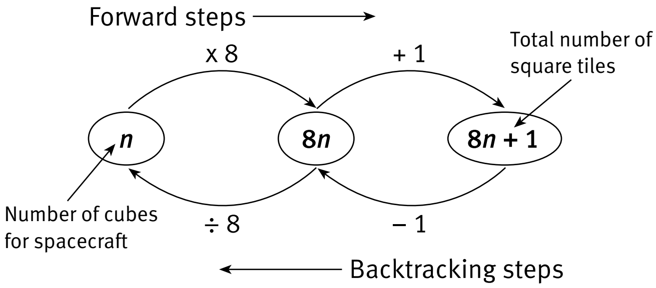


So, for example, in the 3-cube spacecraft, there are 4 lots of 6 tiles plus 1 tile on top, so the calculation is 4 x 6 + 1. In general, for *n* cubes there are 4 x 2*n* + 1 tiles. The rules that develop from the short-cut table shown in the Answers are:

Craft size (number of cubes)	Number of square tiles	
	Jeff's predictions	Fran's predictions
n	1 + 4 x *n* + 4 x *n* = 1 + 4*n* + 4*n*	8 x *n* + 1 or 4 x 2*n* + 1 = 8*n* + 1

As with the previous two rules, Jeff's rule simplifies to Fran's rule, 8*n* + 1.

In question **4d**, the students must work backwards to calculate the number of cubes in a spacecraft with 193 tiles. They will find that using Fran's rule, 8*n* + 1, provides the most straightforward approach. This backtracking flow chart shows how the rule 8*n* + 1 and its reverse work.



Page 14 Domino Stacks

Achievement Objectives

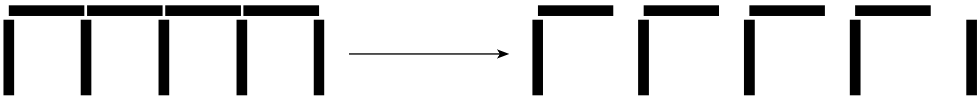
- devise and use problem-solving strategies to explore situations mathematically (Mathematical Processes, problem solving, level 4)
- find a rule to describe any member of a number sequence and express it in words (Algebra, level 4)
- use a rule to make predictions (Algebra, level 4)

ACTIVITY

In this activity, students use number patterns to first devise a rule for the number of dominoes used in the bottom storey of any domino-stack house. They also devise a rule for the total number of dominoes in any domino-stack house.

In question **1**, some students may find it helpful to use dominoes to build houses with 1 storey, then 2, 3, and 4 storeys. Others may find it sufficient to draw diagrams of such a sequence of houses.

The number of dominoes required forms a sequence of odd numbers, so we would expect the bottom storey of a 4-storey house to require 9 dominoes. This is correct, and the following diagram shows how it can be visualised:



There are 2 x 4 + 1 = 9 dominoes in the bottom storey of this 4-storey domino house. A house with *n* storeys will have 2 x *n* + 1 = 2*n* + 1 dominoes. (It is helpful to remember that 2*n* + 1 can be used to generate the odd numbers, 1, 3, 5, 7, … by finding its value when *n* = 0, 1, 2, 3, … respectively.)

Question **1d** requires the students to work backwards and calculate how many storeys high a stack with 69 dominoes in its bottom storey would be. The students might choose to construct a flow chart or to approach the question algebraically: "We know that 2*n* + 1 = 69, so 2*n* = 68 and *n* = 34."

In question **2**, Barbara and Kahu make a table to help figure out a way to predict the total number of dominoes needed for any domino house. A rule for *n* storeys is: *n* x (*n* + 2), which simplifies to *n*(*n* + 2).

There are other rules that some students may suggest, for example, $n^2 + 2n$. Note that $n(n + 2) = n^2 + 2n$, so these two rules are equivalent.

Using $n^2 + 2n$ for a domino house with 13 storeys, we find that it needs 13^2 + 2 x 13 = 13 x 13 + 26
= 169 + 26
= 195 dominoes.

Students may also notice that the numbers, 3, 8, 15, 24, 35, … are all 1 less than the square numbers 4, 9, 16, 25, … respectively. The pattern related to this is shown in the following table:

Number of storeys	Number of dominoes	Pattern in total number of dominoes
1	3	2 x 2 – 1
2	8	3 x 3 – 1
3	15	4 x 4 – 1
4	24	5 x 5 – 1
5	35	6 x 6 – 1
6	48	7 x 7 – 1
n		(*n* + 1) x (*n* + 1) – 1 = $(n + 1)^2 - 1$

So, for example, the number of dominoes for a domino house with 20 storeys is 21 x 21 – 1 = 440. It can also be shown that $(n + 1)^2 - 1$ is equivalent to the rules *n*(*n* + 2) and $n^2 + 2n$.

Question **2d i** is easily solved using the *n* x (*n* + 1) rule. For question **2d ii**, the students are looking for the greatest pair of numbers that are separated by 2 and that multiply to give no more than 4 899. The smaller of the two numbers is *n*, the number of storeys in the house.

The best approach for this question is a trial-and-improvement strategy, starting with a number that is less than the 76 used in **2d i**. Some students may notice that 4 899 is 1 less than 4 900, which is 70^2. This suggests that the pair of numbers should be very close to 70. In fact, they are 69 and 71, which multiply together to give exactly 4 899, the maximum number of available dominoes. So the number of storeys is 69.

Counting Cubes

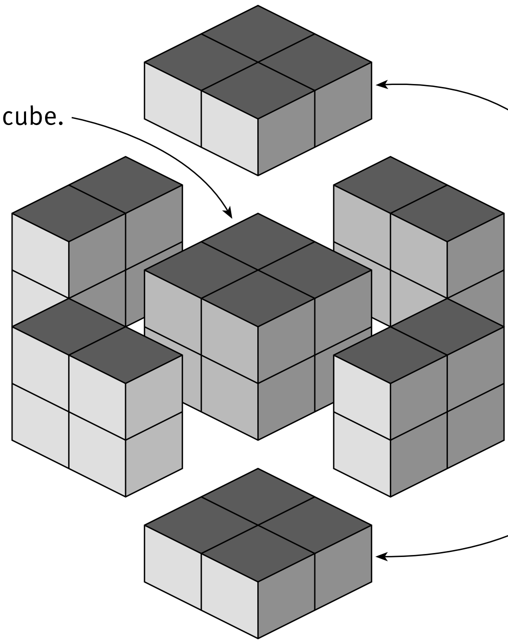
Achievement Objectives

- find and justify a word formula which represents a given practical situation (Algebra, level 4)
- find a rule to describe any member of a number sequence and express it in words (Algebra, level 4)
- use a rule to make predictions (Algebra, level 4)

ACTIVITY

In this activity, students will need to use their spatial skills to visualise the growing patterns of cubes made by Vyshan and Hema. Each of Vyshan's buildings has an inner solid core in the shape of a cube. The core cannot be seen because a layer of cubes covers each of its 6 faces. The following diagram shows the 6 layers for Vyshan's pattern 2.

In the pattern, there are 2 x 2 x 2 = 8 cubes making up the central core. Each of the 6 faces of the core has a single layer of 2 x 2 = 4 cubes attached to it. So pattern 2 has $2^3 + 6 \times 2^2 = 32$ cubes altogether.



The following table shows the number of cubes making up each of Vyshan's patterns and how the patterns in each column lead to algebraic rules.

Pattern number	Number of cubes in core	Number of cubes in layers on six faces of core	Total number of cubes
1	$1 \times 1 \times 1 = 1^3$	$6 \times (1 \times 1) = 6 \times 1^2$	$1^3 + 6 \times 1^2 = 7$
2	$2 \times 2 \times 2 = 2^3$	$6 \times (2 \times 2) = 6 \times 2^2$	$2^3 + 6 \times 2^2 = 32$
3	$3 \times 3 \times 3 = 3^3$	$6 \times (3 \times 3) = 6 \times 3^2$	$3^3 + 6 \times 3^2 = 81$
4	$4 \times 4 \times 4 = 4^3$	$6 \times (4 \times 4) = 6 \times 4^2$	$4^3 + 6 \times 4^2 = 160$
5	$5 \times 5 \times 5 = 5^3$	$6 \times (5 \times 5) = 6 \times 5^2$	$5^3 + 6 \times 5^2 = 275$
n	$n \times n \times n = n^3$	$6 \times (n \times n) = 6 \times n^2 = 6n^2$	$n^3 + 6 \times n^2 = n^3 + 6n^2$

There are various ways of looking at Hema's pattern, but the simplest way to express a rule is explained in the Answers.

The following table shows how this rule works:

Hema's pattern number	Number of cubes
1	12 x 1 + 8 = 20
2	12 x 2 + 8 = 32
3	12 x 3 + 8 = 44
4	12 x 4 + 8 = 56
5	12 x 5 + 8 = 68
8	12 x 8 + 8 = 104
36	12 x 36 + 8 = 440
100	12 x 100 + 8 = 1 208
n	$12n + 8$

Note that the number of cubes for any pattern is 12 more than in the previous pattern because 1 cube is added to each of the 12 edges of a pattern to make the next building in Hema's pattern.

In answer to question **3a**, students with a good spatial sense may notice that there is a simple connection between Vyshan's and Hema's buildings. For example, Vyshan's first pattern fits into Hema's first pattern to make a 3 x 3 x 3 = 27 solid cube. Their second patterns fit together to make a 4 x 4 x 4 = 64 solid cube, and so on. Geometrically, we say that the patterns complement one another.

The following table shows how this relationship can be used. Note that the number of cubes on each edge is 2 more than the pattern number. (So in pattern 1, there are 1 + 2 = 3 cubes on each edge.)

Pattern number	Total number of cubes in joined patterns	Number of cubes in Hema's patterns	Number of cubes in Vyshan's patterns
1	$(1 + 2)^3 = 3^3$ $= 27$	12 x 1 + 8 = 20	27 – 20 = 7
2	$(2 + 2)^3 = 4^3$ $= 64$	12 x 2 + 8 = 32	64 – 32 = 32
3	$(3 + 2)^3 = 5^3$ $= 125$	12 x 3 + 8 = 44	125 – 44 = 81
4	$(4 + 2)^3 = 6^3$ $= 216$	12 x 4 + 8 = 56	216 – 56 = 160
n	$(n + 2)^3$	$12n + 8$	$(n + 2)^3 - (12n + 8)$

If we know any two of the column entries for a given pattern, we can easily calculate the third. For example, the number of cubes in each of Vyshan's patterns can be found by subtracting the numbers of cubes in Hema's corresponding pattern from the total number of cubes in the joined patterns.

From One to Another

Achievement Objectives

- use words and symbols to describe and generalise patterns (Mathematical Processes, developing logic and reasoning, level 4)
- sketch and interpret graphs on whole number grids which represent simple everyday situations (Algebra, level 4)

ACTIVITY

In this activity, the students first use newspaper to make a square measuring 1 metre by 1 metre. It is important that the students make reasonably accurate measurements (for instance, to the nearest centimetre) and that they are comfortable in making conversions between units. For example, they will need to convert measurements in centimetres or millimetres to their equivalents in metres in order to plot them on the graph. They also need to check that each of their rectangles uses all of the newspaper in the square that they first make. This will ensure that the square and each rectangle has an area of 1 square metre (1 m^2).

The students may have some difficulty plotting points representing the measurements of their rectangles. The scale on any graph they make is unlikely to accommodate measurements to the nearest millimetre or even centimetre. Unless they use a computer to draw the graph, the best they are likely to achieve is a scale showing measurements to the nearest 10 centimetres. This is enough to show the shape of the curve and to appreciate what is happening mathematically.

The graph in the Answers for **2a** shows the points representing the dimensions for several rectangles. In fact, any point on this curve represents the dimensions of a rectangle with an area of 1 square metre.

The point (1,1) represents the square measuring 1 metre by 1 metre. In order to clearly see the pattern made by the points, the students will need to join the points with a smooth curve. (It's often best to use a pencil for this.) The curve is known as a hyperbola. Notice that it gets closer and closer to the two axes but never touches them. Any rectangle they make has a finite area because there is always a (non-zero) length and width.

Note that it is possible to have either of two points to represent a rectangle that has the dimensions 20 centimetres (or 0.2 metres) and 5 metres. One point is C (0.2,5), and the other is D (5,0.2). The first co-ordinate given is, by convention, the dimension for the horizontal axis (in this case, length) and the second is for the vertical axis (width).

Rectangle measurements that are represented by identified points on the hyperbola are shown in the following table:

Area (square m)	Length (m)	Width (m)
1	0.20	5.00
1	0.25	4.00
1	$0.3\dot{3}$	3.00
1	0.50	2.00
1	1.00	1.00
1	2.00	0.50
1	3.00	$0.3\dot{3}$
1	4.00	0.25
1	5.00	0.20
$x \times y$	x	y

Notice that measurements for length and width are rounded to the nearest centimetre (hundredths of a metre). For example, a measurement of 25 centimetres is represented in the table above as 0.25 metres.

The rule for the pattern is $1 = x \times y$, which is usually written as $1 = xy$ or $xy = 1$. It can also be written as $y = \frac{1}{x}$ or even $x = \frac{1}{y}$. The students might like to see if they can calculate how wide a rectangle with an area of 1 m^2 would be if its length were 1 kilometre. Using the relationships described, a rectangle with an area of 1 m^2 and length $x = 1$ kilometre (1 000 metres) will have width $y = \frac{1}{1\,000}$, or 0.001 metres. So the width of this rectangle is 0.001 metres = 1 millimetre.

Page 17

Areas of Interest

Achievement Objectives

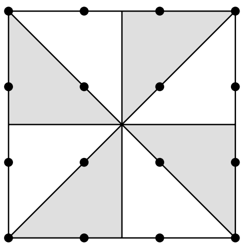
- find and justify a word formula which represents a given practical situation (Algebra, level 4)
- find a rule to describe any member of a number sequence and express it in words (Algebra, level 4)
- use a rule to make predictions (Algebra, level 4)

ACTIVITY

In this activity, the students must devise ways to calculate the shaded area for each pattern shown. They may attempt to calculate the shaded area directly or they may consider the non-shaded area and then subtract this from the total area shown. In order to find a rule for any pattern in a particular sequence, the students will find it helpful to record the areas in a table. The comments below on each section of question **1c** will help the students with the areas and rules in questions **1a** and **1b**.

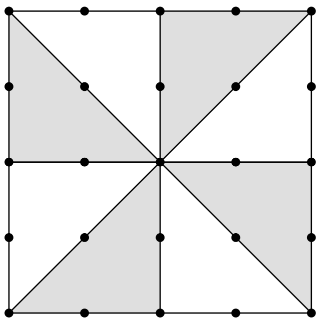
In question **1c i**, the shaded area for each pattern is half the total area enclosed in the pattern. For example, the total area enclosed for pattern 2 below is 3 units by 3 units, or 9 units^2. The shaded area is half of the total area enclosed, that is, $^1/_2$ of 9 units^2 = 4.5 units^2. Similarly, the shaded area for pattern 3 is $^1/_2$ of 16 units^2 = 8 units^2.

Total area = 9 units^2
Shaded area = 4.5 units^2



Pattern 2

Total area = 16 units^2
Shaded area = 8 units^2



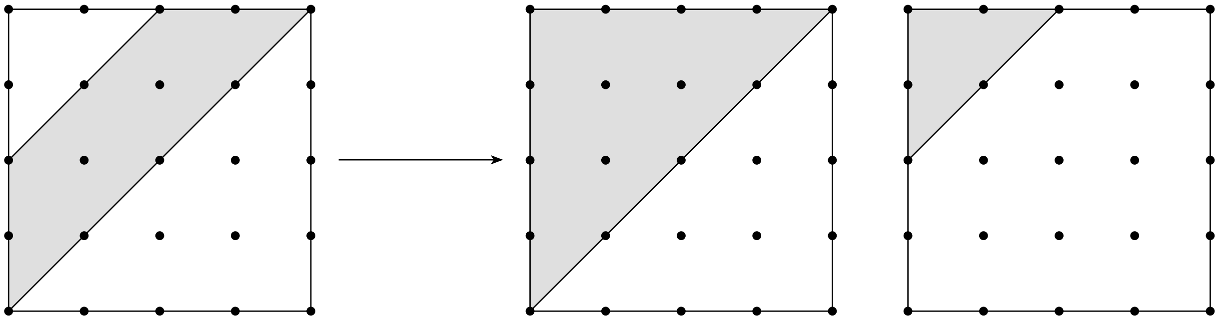
Pattern 3

The following table shows the shaded areas and number patterns for the sequence of shaded areas in question **1c i**.

Pattern number	Calculation of area	Area (units^2)
1	$^1/_2$ of 2 x 2	2
2	$^1/_2$ of 3 x 3	4.5
3	$^1/_2$ of 4 x 4	8
4	$^1/_2$ of 5 x 5	12.5
5	$^1/_2$ of 6 x 6	18
10	$^1/_2$ of 11 x 11	60.5
n	$^1/_2$ of $(n + 1) \times (n + 1)$	$^1/_2(n + 1)^2$

So, for example, the shaded area in the hundredth pattern is $^1/_2$ of 101 x 101 = 5 100.5 units^2.

Pattern 2 for question **1c ii** is shown below. There are several approaches that might be taken to this question. One approach is to analyse the shaded area in the way shown below.



$^1/_2$ of 4 x 4 = 8 units^2 $^1/_2$ of 2 x 2 = 2 units^2

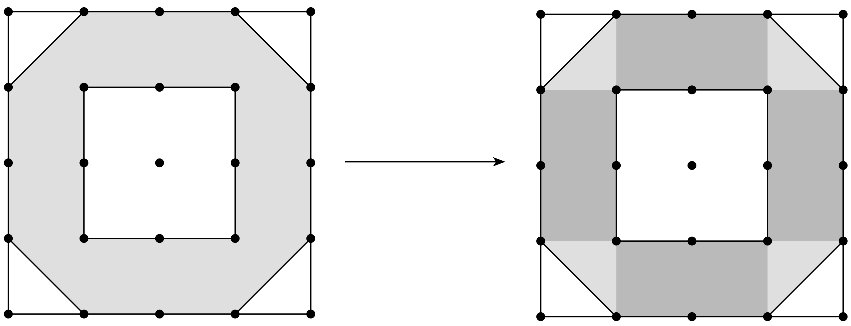
The shaded area can be thought of as the difference between the highlighted area in the second diagram and the highlighted area in the third diagram. Note that the highlighted area in the third diagram ($^1/_2$ of 2 x 2 = 2 units^2) is the same for each pattern in question **ii**.

The shaded area for pattern 2 = $^1/_2$ x (4 x 4) – 2 units^2
= $^1/_2$ x 16 – 2
= 6 units^2

The associated table is:

Pattern number	Calculation of area	Area (units^2)
1	$^1/_2$ of (3 x 3) – 2	2.5
2	$^1/_2$ of (4 x 4) – 2	6
3	$^1/_2$ of (5 x 5) – 2	10.5
4	$^1/_2$ of (6 x 6) – 2	16
5	$^1/_2$ of (7 x 7) – 2	22.5
10	$^1/_2$ of (12 x 12) – 2	70
n	$^1/_2$ of $(n + 1) \times (n + 1)$	$^1/_2(n + 1)^2$

Pattern 2 for question **1c iii** is shown below.

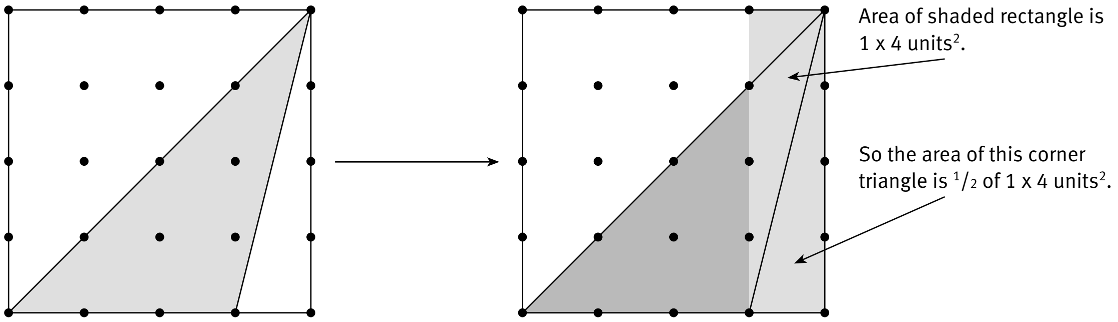


The shaded area for this pattern can be considered as consisting of 4 lots of 2 units^2 (2 on each side) plus the 4 triangles that link these sides. The calculation is then 4 x 2 + 4 x $^1/_2$ = 4 x 2 + 2 units^2. The associated table is:

Pattern number	Calculation of area	Area (units^2)
1	4 x 1 + 2	6
2	4 x 2 + 2	10
3	4 x 3 + 2	14
4	4 x 4 + 2	18
5	4 x 5 + 2	22
10	4 x 10 + 2	42
n	4 x *n* + 2	$4n + 2$

So, for example, the shaded area in the hundredth pattern is 4 x 100 + 2 = 402 units^2.

Pattern 3 for question **1c iv** is shown below.



The shaded area for pattern 3 is $^1/_2$ of 4 x 4 – $^1/_2$ of 1 x 4 = $^1/_2$ of ((4 x 4) – (1 x 4))
= $^1/_2$ of (3 x 4)
= $^1/_2$ of 12
= 6 units^2.

The associated table is:

Pattern number	Calculation of area	Area (units^2)
1	$^1/_2$ of 1 x 2	1
2	$^1/_2$ of 2 x 3	3
3	$^1/_2$ of 3 x 4	6
4	$^1/_2$ of 4 x 5	10
5	$^1/_2$ of 5 x 6	15
10	$^1/_2$ of 10 x 11	55
n	$^1/_2$ of n x (n + 1)	$^1/_2n(n + 1)$

So, for example, the shaded area for the hundredth pattern is $^1/_2$ of 100 x (100 + 1) = $^1/_2$ x 100 x 101
= 5 050 units^2

Note that we might also approach question **1c iv** in the following way: If n is the pattern number, then the area of the narrow unshaded triangle is $^1/_2$ x (1 x (n + 1)) = $^1/_2$ x (n + 1)

$$\frac{n+1}{2} = \text{units}^2.$$

(For example, in pattern 3, the area of this triangle is $\frac{3+1}{2}$ = 2). So an alternative formula for calculating the total area shaded would be $^1/_2(n + 1)^2 - {}^1/_2(n + 1) = {}^1/_2((n + 1)^2 - (n + 1))$. It can be shown that this equation is equivalent to $^1/_2n(n + 1)$:

$^1/_2(n + 1)^2 - {}^1/_2(n + 1)$
$= {}^1/_2((n + 1)^2 - (n + 1))$
$= {}^1/_2(n^2 + 2n + 1 - n - 1)$
$= {}^1/_2(n^2 + n)$
$= {}^1/_2n(n + 1)$

Pages 18–19 Mats, Patterns, and Rules

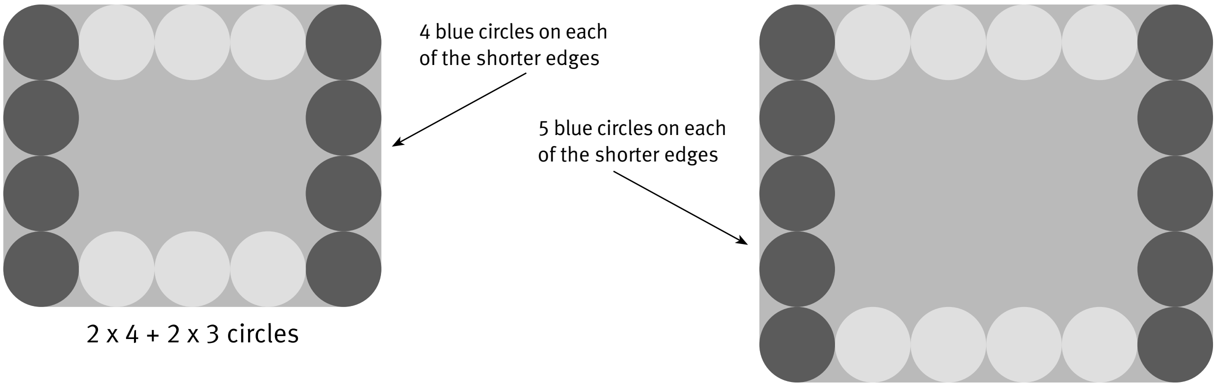
Achievement Objectives

- find and justify a word formula which represents a given practical situation (Algebra, level 4)
- find a rule to describe any member of a number sequence and express it in words (Algebra, level 4)
- use a rule to make predictions (Algebra, level 4)

ACTIVITY

In this activity, students devise rules to predict the number of circles on the borders of table mats. In each of the different designs, the longer edge always has 1 more circle than the shorter edge.

In question **1**, the students need to use the pattern of coloured circles in Evalesi's designs to help explain the rule. The two short cuts for calculating the number of circles in the table mats shown are:



2 x 4 + 2 x 3 circles

2 x 5 + 2 x 4 circles

The pattern in these two short cuts suggests that a design with 6 circles on the shorter edge will have 2 sets of 6 blue circles and 2 sets of 5 yellow circles, that is, 2 x 6 + 2 x 5 = 22 circles altogether. A table showing how Amber's rule can be derived from the short cuts for Evalesi's designs is given in the Answers. Looking at the calculations in this table, it is clear that the x in Amber's rule is the number of circles on the shorter edge of the table mat. The last row in the table below shows how the rule is formed.

Evalesi's Table Mat Designs			
Number of circles on shorter side	Number of blue circles	Number of yellow circles	Total number of circles
3	2 x 3	2 x 2	2 x 3 + 2 x 2 = 10
⋮	⋮	⋮	⋮
x	$2 \times x$ $= 2x$	$2 \times (x - 1)$ $= 2(x - 1)$	$2 \times x + 2 \times (x - 1)$ $= 2x + 2(x - 1)$

The short cuts leading to the algebraic rules for the designs in question **2** are shown in the tables below.

Terri's Table Mat Designs				
Number of circles on shorter side	Number of:			Total number of circles
	blue circles	orange circles	purple circles	
3	2 x 2	2 x 2	2	4 x 2 + 2 = 10
4	2 x 3	2 x 3	2	4 x 3 + 2 = 14
5	2 x 4	2 x 4	2	4 x 4 + 2 = 18
6	2 x 5	2 x 5	2	4 x 5 + 2 = 22
10	2 x 9	2 x 9	2	4 x 9 + 2 = 38
x	$2 \times (x - 1)$ $= 2(x - 1)$	$2 \times (x - 1)$ $= 2(x - 1)$	2	$2 \times (x - 1) + 2 \times (x - 1) + 2$ $= 2(x - 1) + 2(x - 1) + 2$ $= 4(x - 1) + 2$

Waione’s Table Mat Designs				
Number of circles on shorter side	Number of: yellow circles	orange circles	purple circles	Total number of circles
3	2 x 1	2 x 2	4	2 x 1 + 2 x 2 + 4 = 10
4	2 x 2	2 x 3	4	2 x 2 + 2 x 3 + 4 = 14
5	2 x 3	2 x 4	4	2 x 3 + 2 x 4 + 4 = 18
6	2 x 4	2 x 5	4	2 x 4 + 2 x 5 + 4 = 22
10	2 x 8	2 x 9	4	2 x 8 + 2 x 9 + 4 = 38
x	2 x $(x - 2)$ = $2(x - 2)$	2 x $(x - 1)$ = $2(x - 1)$	4	2 x $(x - 2)$ + 2 x $(x - 1) + 4$ = $2(x - 2) + 2(x - 1) + 4$ = $4x + 1$

In question **2c**, the formula in cell B2 is =4*(A2–1)+2. The value in cell A2 is 5, so the formula calculates 4 x (5 – 1) + 2 = 18. In cell B3, the formula will be =4*(A3–1)+2, which calculates 4 x (4 – 1) + 2 = 14. And in cell B9, the formula =4*(A9–1)+2 calculates 4 x (1 000 – 1) + 2 = 3 998 because the value in cell A9 is 1 000. The formulae in the active cells in column B can all be represented by the algebraic rule $4(x - 1) + 2$, where x stands for the number of circles on the shorter side of any table mat.

The rule that describes Waione’s design pattern is $2(x - 2) + 2(x - 1) + 4$. So the formula that goes in cell C2 is =2*(A2–2)+2*(A2–1)+4. The formula that goes in cell C3 will be =2*(A3–2)+2*(A3–1)+4 and, in cell C9, the formula will be =2*(A9–2)+2*(A9–1)+4. The value in cell A9 is 1 000, so the formula in cell C9 calculates 2 x 998 + 2 x 999 + 4 = 3 998. Note that this value is the same as that calculated for cell B9 above.

The spreadsheet below shows the formulae for the spreadsheet for question **2c iii**. The spreadsheet given in the Answers shows the values calculated by the formulae.

Table Mats: Terri and Waione (SS)

D10

	A	B	C
1	Number of circles on shortest side	Short cut for Terri’s design	Short cut for Waione’s design
2	5	=4*(A2–1)+2	=2*(A2–2)+2*(A2–1)+4
3	4	=4*(A3–1)+2	=2*(A3–2)+2*(A3–1)+4
4	7	=4*(A4–1)+2	=2*(A4–2)+2*(A4–1)+4
5	10	=4*(A5–1)+2	=2*(A5–2)+2*(A5–1)+4
6	27	=4*(A6–1)+2	=2*(A6–2)+2*(A6–1)+4
7	186	=4*(A7–1)+2	=2*(A7–2)+2*(A7–1)+4
8	253	=4*(A8–1)+2	=2*(A8–2)+2*(A8–1)+4
9	1000	=4*(A9–1)+2	=2*(A9–2)+2*(A9–1)+4

In question **3**, Evalesi notices that each short-cut answer is 4 times the number of circles on the shortest side minus 2. Evalesi’s rule is therefore 4 x $x - 2$, or $4x - 2$. By choosing any value of x from the spreadsheet, the students will find that $\square$ = 2. (The calculations for the case in which $x = 7$ are given in the Answers.)

Note that the algebraic rules given earlier for the designs for Evalesi, Terri, and Waione all simplify to $4x - 2$, as shown in the Answers.

Pages 20–21 **Rotten Apples**

Achievement Objectives

- find and justify a word formula which represents a given practical situation (Algebra, level 4)
- find a rule to describe any member of a number sequence and express it in words (Algebra, level 4)
- use a rule to make predictions (Algebra, level 4)

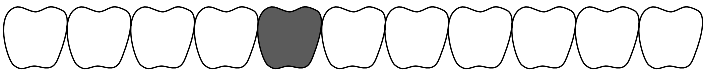
ACTIVITY

In this activity, students investigate the effect of having a rotten apple in different-sized trays of apples. Apples in contact with a rotten apple become rotten themselves after 1 week.

In question **1a**, 11 apples are packed in the tray. The apple in the first space was rotten when it was packed. At the end of each successive week, 1 more apple is rotten. After 10 weeks, there are 1 + 10 rotten apples. After *n* weeks, there will be 1 + *n* rotten apples.

In question **1b**, the position of the initial rotten apple in trays A to E has been systematically changed to encourage the students to tackle the questions in a similar systematic way.

One way to approach this question is to consider the number of apples on either side of the rotten apple. For example, in tray B, we have 4 apples to the left of the rotten apple and 6 apples to the right:



In every tray, we can concentrate on the time it takes for the *greater* number of apples (on whichever side of the rotten apple this number occurs) to become rotten because the smaller number of apples will certainly also become rotten in this time.

So, using the results from question **1a**, we can see that in tray B, it will take 6 weeks for all the apples to become rotten. A similar analysis can be used for each of the other trays.

Having considered the times taken for all of the apples to go rotten in the trays in question **1b**, the students should see that the maximum time to rot will occur when the rotten apple is placed at one end of the tray. For an 11-apple tray, this will be 10 weeks (as in question **1a**).

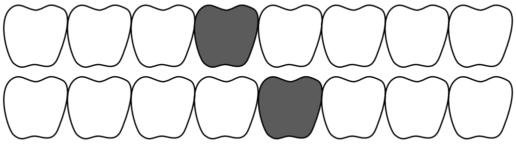
Telea then forms the rule: the maximum number of weeks it takes all the apples to become rotten is 1 fewer than the number of apples in the tray. Kathy, by letting *n* equal the number of apples in the tray, forms the algebraic rule: the maximum number of weeks = $n - 1$.

In question **1d**, Kathy finds a rule for the minimum number of weeks it will take for all the apples in the tray to become rotten. She notices that this occurs when the rotten apple is packed in the middle of the tray, so there is an equal number of apples on either side of the rotten apple. When there are 11 apples, the minimum time, 5 weeks, will occur when there are 5 apples on each side of the rotten apple. The following table shows the number of weeks it takes for all apples to become rotten when there is an odd number of apples packed in a row and the rotten apple has been packed in the centre space.

Number (odd) of apples in a single-row tray	Number of weeks in tray for all apples to become rotten
3	$(3 - 1) \div 2 = 1$
5	$(5 - 1) \div 2 = 2$
7	$(7 - 1) \div 2 = 3$
9	$(9 - 1) \div 2 = 4$
11	$(11 - 1) \div 2 = 5$
27	$(27 - 1) \div 2 = 13$
n	$(n - 1) \div 2 = \frac{n-1}{2}$

When there is an odd number, n, of apples in a row, there are $\frac{n-1}{2}$ apples on either side of the middle apple. One apple on each side of the middle apple becomes rotten each week, so it takes $\frac{n-1}{2}$ weeks for $\frac{n-1}{2}$ apples on either side of the middle apple to become rotten.

The following table shows the minimum time for an even-numbered tray of apples to become rotten. This will occur when the rotten apple is packed as close to the middle as possible. So, for example, if there are 8 apples in a tray, the minimum time for all 8 apples to become rotten will occur when the rotten apple is packed in either the fourth or the fifth position in the tray.



Number (even) of apples in a single-row tray	Greatest number of apples on either side of rotten apple	Number of weeks in tray for all apples to become rotten
4	$4 \div 2 = 2$	2
6	$6 \div 2 = 3$	3
10	$10 \div 2 = 5$	5
20	$20 \div 2 = 10$	10
100	$100 \div 2 = 50$	50
n	$n \div 2$	$n \div 2 = \frac{n}{2}$

Once again, we are interested in the greatest number of apples on either side of the rotten apple, which is $n \div 2$ or $\frac{n}{2}$. So the minimum time that an even number (n) of apples in a row takes to become rotten is $\frac{n}{2}$ weeks.

The rules above for the maximum and minimum times that a row of apples in a tray takes to become rotten can also be used to work out the minimum and maximum numbers of apples possible in trays where the apples all become rotten in a given number of weeks.

Suppose the apples all become rotten in n weeks. The minimum number of apples in the tray will then be $n + 1$, that is, 1 more than the number of weeks. This occurs when the rotten apple is packed in the first (or last) space in the row. The maximum number of apples that can become rotten in n weeks occurs when there are n apples on either side of the middle rotten apple. Then there are $2n + 1$ apples altogether in the tray. For example, when all the apples become rotten after 27 weeks, the minimum number of apples possible is $27 + 1 = 28$ and the maximum number of apples possible is $2 \times 27 + 1 = 55$.

In question **2**, the tray used for packing apples is more realistic. It is square-shaped with 11 x 11 = 121 spaces for apples. The students draw diagrams to show how many apples become rotten after 3 and also 4 weeks. They should notice a sequence, 4, 8, 12, 16, … , for the numbers of new rotten apples after 1, 2, 3, 4, … weeks respectively. They then use a spreadsheet to calculate the total number of apples that have become rotten after each week. They can use Kathy's formula (4 x *n*) for working out the values in column B. For example, the formula in cell B3 might read =4*A3. Note that the value 1 in cell B2 does *not* follow this rule.

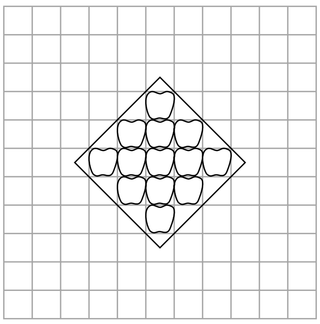
Kathy's Spreadsheet (SS)

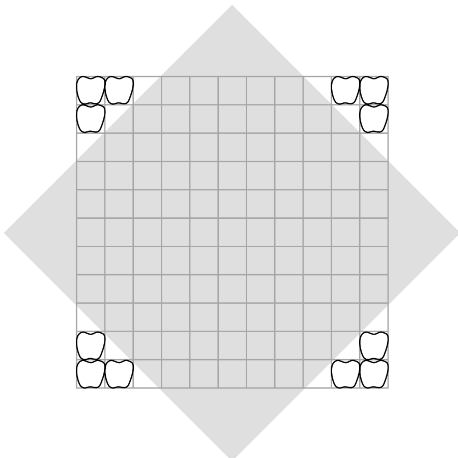
C14 =C13+B14

	A	B	C
1	Week	Number of new rotten apples	Total number of rotten apples
2	0	1	1
3	1	4	5
4	2	8	13
5	3	12	25
6	4	16	41
7	5	20	61
8	6	24	85
9	7	28	113
10	8	32	145
11	9	36	181
12	10	40	221
13	11	44	265
14	12	48	313

The formula =C2+B3 calculates the value for the total number of rotten apples after 1 week, =C3+B4 calculates the value for the total number of rotten apples after 2 weeks, and so on. Kathy notices that her spreadsheet indicates that 113 apples will be rotten after 7 weeks and 145 apples will be rotten after 8 weeks. So she claims (wrongly) that all 121 apples in an 11 by 11 tray will become rotten after 8 weeks.

Kathy is incorrect because although her spreadsheet accurately counts the number of rotten apples after *n* weeks, it does so only for the particular diamond pattern of apples shown in question **2**. Kathy's mistake is to assume that all of the 121 apples shown in the square tray would fit within a diamond pattern of 145 apples; in fact, they would not.





Telea makes his claim of 5 weeks by using knowledge gained from the earlier work with apples in rows. He considers how many weeks it would take for the rotten apples to reach the edge of the tray and correctly calculates this to be 5 weeks. Like Kathy, however, Telea's calculation for the time taken for all 121 apples in the tray to become rotten is incorrect because he also ignores the apples becoming rotten in the diagonals of the tray. The diagram in the Answers shows what really happens with the apples in a square-shaped tray.

Note that the corner apple becomes rotten after 10 weeks. So, as we have seen, neither Kathy nor Telea is correct.

The students might like to confirm this result by drawing a series of 11 by 11 grid diagrams that show the results after 5, 8, and 10 weeks. Having done this, they might also like to reflect on the fact that, after 10 weeks, the 121 rotten apples in the 11 by 11 tray would have grown to 221 rotten apples (as in the spreadsheet on the previous page) in a diamond pattern, had the tray been large enough to accommodate them.

Pages 22–23 Suspended Thought

Achievement Objectives

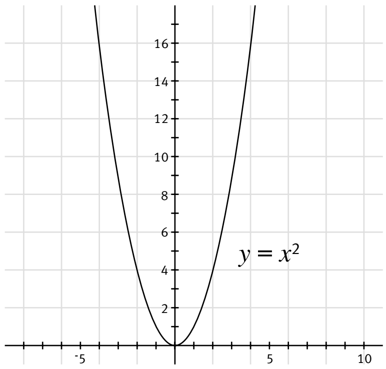
- sketch and interpret graphs on whole number grids that represent simple everyday situations (Algebra, level 4)
- find, and use with justification, a mathematical model as a problem-solving strategy (Mathematical Processes, problem solving, level 4)
- interpret information and results in context (Mathematical Processes, developing logic and reasoning, level 4)

ACTIVITY

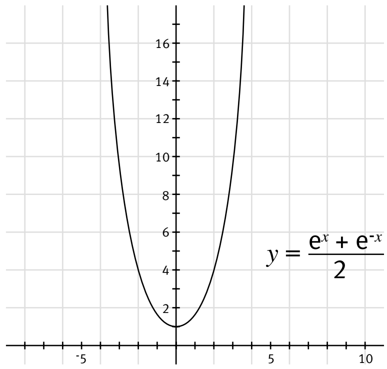
In this activity, students carry out a practical experiment to investigate aspects of the mathematics of the curves found in suspension bridges.

The students model the behaviour of the cable of a suspension bridge and measure the "sag" at a number of points spaced along the curve. They use their results to estimate the length of the suspender cables needed for a real bridge.

Over the years, many famous mathematicians have contributed to our understanding of the "chain-curves" formed when chains, ropes, or cables hang freely from their ends. Galilieo Gallilei (1564–1642) claimed wrongly that the curve is a parabola, rather like an upside-down version of the curve associated with a path of a ball thrown from one person to another. In 1691, mathematicians Gottfried Wilhelm von Leibniz (1646–1716), Christian Huygens (1629–1695), and Johann Bernouilli (1667–1748) found the equation of the chain curve that they named a catenary, from the Latin word for chain. Graphs of a parabola and a catenary are shown on the following page so that they can be compared.



Parabola



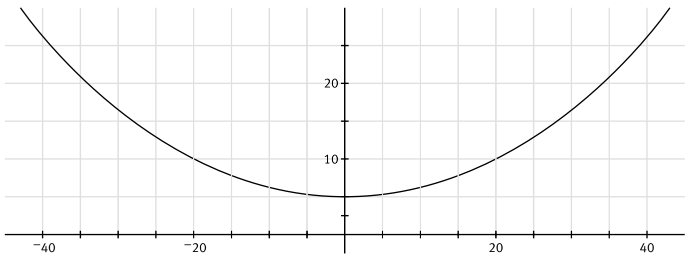
Catenary

Notice the differences between the two shapes. The more rounded catenary is due to the influence of gravity on each point of the curve and the tension in the chain, rope, or cable used for the catenary. The basic equation for the catenary is: $y = \frac{e^x + e^{-x}}{2}$. (The students are not expected to use this sort of algebraic equation at this stage.)

In this equation, *e* represents the constant value 2.718 281 828 …, known as "exponential *e*". Here it is expressed to 9 decimal places but, in fact, it is an irrational number (a non-terminating, non-recurring decimal, like π [pi]). It pops up regularly in the mathematics associated with naturally occurring situations, especially in engineering. While the catenary in this activity relates to suspension bridges, the same ideas can also be applied to arches. The catenary arch is the arch that makes the best possible use of gravity to support itself. For an excellent website on this subject, designed for inquiring school students, see www.cpo.com/Weblabs/chap3/archf.htm

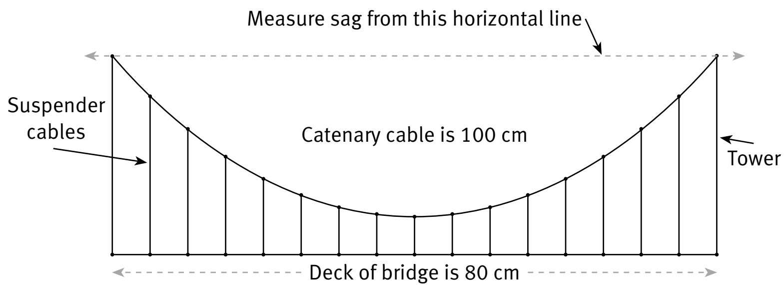
The cable catenaries that we see in suspension bridges appear more stretched than the catenary shown above. The curve below more accurately reflects the shape of the rope catenary that the students make with their model.

Its formula is $y = \frac{40(e^{\frac{x}{40}} + e^{\frac{-x}{40}})}{2} - 35$.



On their model, the students measure the sag at 5 centimetre intervals and then work out the lengths of the suspenders (vertical cables) measured from the catenary cable to the deck of the bridge.

The following diagram shows the measurements that the students should make.



The students use the results from their experimental work to try to predict the length of the suspenders for a real suspension bridge.

The first investigation asks why the sides of a suspended rope or chain hang more steeply than the centre. This can be answered by imagining a chain with 200 links suspended over a gap. The two end links are each supporting the 99 links below them. The second-from-top links are supporting the 98 links below them, and so on. But the two links at the bottom are supporting nothing; they are simply "hanging on". The links closer to the top of the sides are supporting the greatest mass, so the effect of gravity is felt most strongly by them, hence their more vertical tendency.

The second investigation invites students to find out about the Golden Gate Bridge. Built in 1937, this is the most famous of all suspension bridges, and it is still remarkable for its size, its engineering, and its beauty. Many consider that it is one of the top 10 engineering triumphs of the twentieth century. The facts about its construction and the materials that went into its making should interest students. Good websites with information and photos are easily found. Some of these are:
www.goldengatebridge.org/research/construction.html
www.danheller.com/sf-ggbridge.html
www.sfmuseum.org/hist9/mcgloin.html
www.orbitforkids.com/yellow/dt_yellow_golden.html

A third investigation would be to look into the advantages that suspension bridges have over swing bridges. Three significant advantages are:
- they can span much greater gaps
- they can carry much greater loads
- they can have flat decks.

Page 24 Jam Jars

Achievement Objectives

- sketch and interpret graphs on whole number grids which represent simple everyday situations (Algebra, level 4)
- use their own language, and mathematical language and diagrams, to explain mathematical ideas (Mathematical Processes, communicating mathematical ideas, level 4)

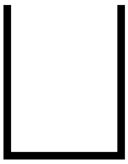
ACTIVITY

In this activity, the students match graphs with the objects (jam jars) that are most likely to have led to the graphical representations.

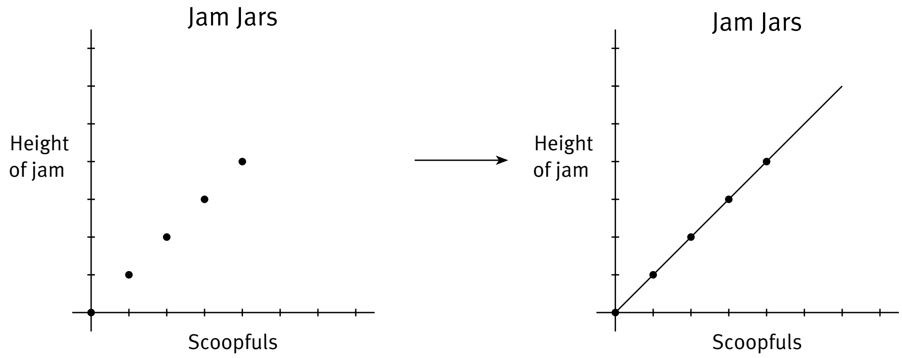
In each case, they must try to visualise the rate at which the height of the jam changes as each scoopful is added. In some of the later examples, this can be quite challenging, and the students should be encouraged to "read" the graphs for clues. For example, the vertical axis measures the height of the jam in the jar. When the jar is full, the line stops, and therefore this point marks the height of the jar. It follows that the relative heights of the jars should be reflected in the relative heights of the graphs for these jars.

There is a further subtlety that the students need to grasp. They may be tempted to conclude that the height of the graph reflects the total volume of the jar and to use this idea to help match jars to graphs. In fact, as we have seen, the height of the graph measures the height of the jar. The volume of the jar is measured by the number of scoops of jam it contains, so the further a graph extends on the *horizontal* axis, the greater the volume of the jar.

In order to set "benchmarks" against which the graphs for different jars can be compared, students might begin by exploring what happens with three particular jars. The simplest type is a cylinder:



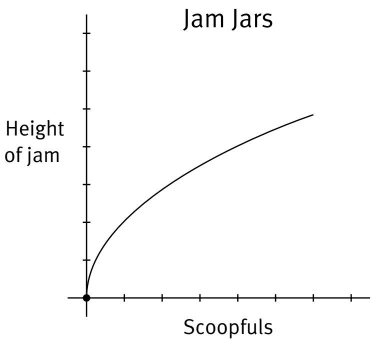
Cylindrical jar



The first graph shows how the height increases as each scoopful is added to the jar. The second graph is made by joining the points in the first graph. The graph is a straight line that shows the height increasing at a steady or constant rate.

While this is not actually the case (the jars are being filled a scoop at a time), we can model the situation in this way without losing any important information. This is a common event in mathematical modelling, where we often simplify in order to assist our calculations or understanding.

The second and third jars that might be investigated are cone shaped. Figure 1 below shows a jar that is part of a cone. It is narrower at the bottom than at the top. Figure 2 shows part of a cone that is narrower at the top than at the bottom. Their graphs are shown below.



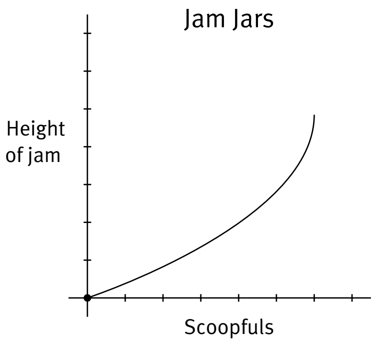


Figure 1

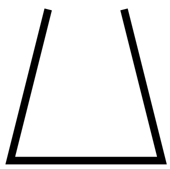


Figure 2

For the jar in figure 1, the rate at which the height of the jam increases gradually slows down. So the graph curves "down". For the jar in figure 2, the rate at which the height of the jam increases gradually speeds up. So the graph for this jar curves "up".

Note that the jars have the same height and the same volume (one is simply the other turned upside down), so the graphs for the two jars finish at the same height and extend the same distance to the right. In fact, the curves are identical if rotated through 180 degrees around a point halfway between the start and end points of the curve.

The shapes of the jars in questions **1** and **2** are a mix of cylinders and parts of cones. The graphs of these "composite" containers will therefore be a mix of the three graphs above. The students will find it very helpful to describe how the height changes before they begin drawing graphs for question **2**. For example, in question **2b**, they might say: "The height increases steadily for the part of the jar that is cylindrical and then speeds up for the cone-shaped part. So the first part of the graph will be a straight line, and in the second part, the line will curve upwards."

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