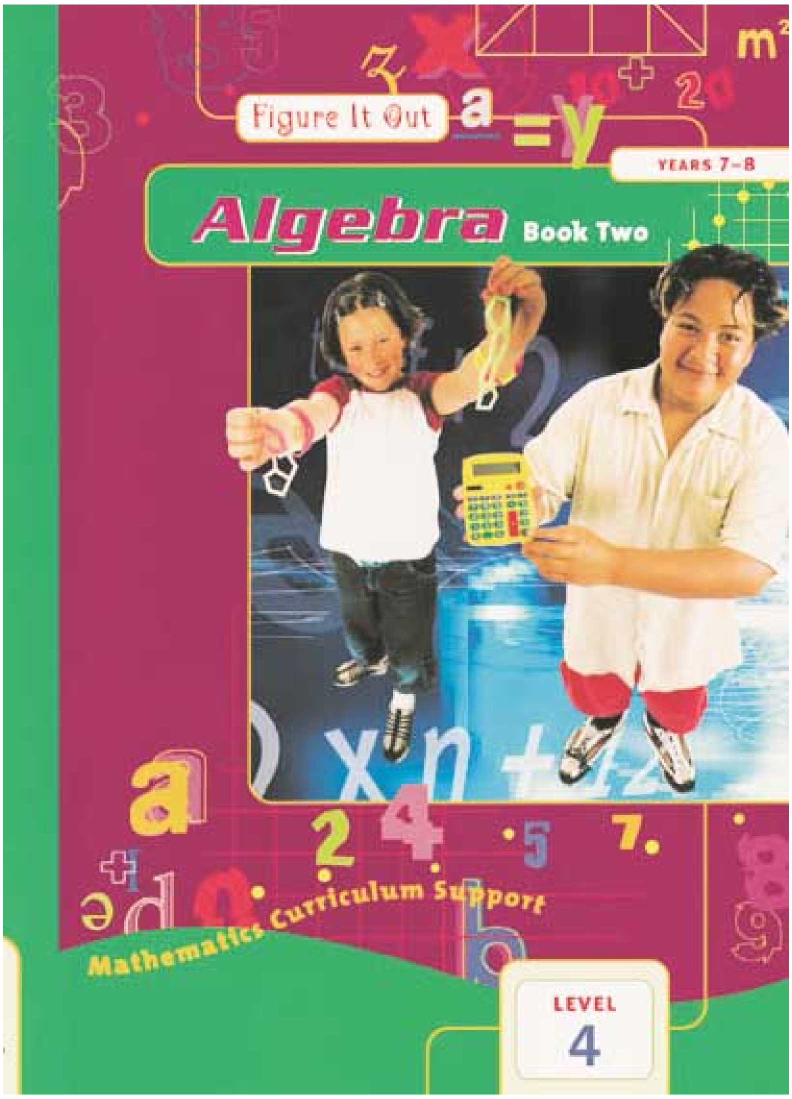
Answers and Teachers' Notes



MINISTRY OF EDUCATION
Te Tāhuhu o te Mātauranga

CONTENTS

The books for years 7–8 in the Figure It Out series are issued by the Ministry of Education to provide support material for use in New Zealand year 7–8 classrooms. The books have been developed and trialled by classroom teachers and mathematics educators and follow on from the successful series for levels 2–4 in primary schools.

Student books

The student books in the series are divided into three curriculum levels: levels 2–3 (linking material), level 4, and level 4+ (extension material). All the books are aimed at year 7–8 students in terms of context and presentation.

The following books are included in the series:
Number (two linking, three level 4, one level 4+, distributed in November 2002)
Number Sense (one linking, one level 4, distributed in April 2003)
Algebra (one linking, two level 4, one level 4+, distributed in August 2003)
Geometry (one level 4, one level 4+, distributed in term 1 2004)
Measurement (one level 4, one level 4+, distributed in term 1 2004)
Statistics (one level 4, one level 4+, distributed in term 1 2004)
Themes: *Disasters Strike!, Getting Around* (level 4–4+, distributed in August 2003)

The activities in the student books are set in meaningful contexts, including real-life and imaginary scenarios. The books have been written for New Zealand students, and the contexts reflect their ethnic and cultural diversity and life experiences that are meaningful to students aged 11–13 years. The activities can be used as the focus for teacher-led lessons, as independent bookwork, or as the catalyst for problem solving in groups.

Answers and Teachers' Notes

The Answers section of the *Answers and Teachers' Notes* that accompany each of the student books includes full answers and explanatory notes. Students can use them for self-marking, or you can use them for teacher-directed marking. The teachers' notes for each activity, game, or investigation include relevant achievement objectives, comment on mathematical ideas, processes, and principles, and suggestions on teaching approaches. The *Answers and Teachers' Notes* are also available on Te Kete Ipurangi (TKI) at www.tki.org.nz/r/maths/curriculum/figure

Using Figure It Out in your classroom

Where applicable, each page starts with a list of equipment that the students will need to do the activities. Encourage the students to be responsible for collecting the equipment they need and returning it at the end of the session.

Many of the activities suggest different ways of recording the solution to a problem. Encourage your students to write down as much as they can about how they did investigations or found solutions, including drawing diagrams. Discussion and oral presentation of answers is encouraged in many activities, and you may wish to ask the students to do this even where the suggested instruction is to write down the answer.

> The ability to communicate findings and explanations, and the ability to work satisfactorily in team projects, have also been highlighted as important outcomes for education. Mathematics education provides many opportunities for students to develop communication skills and to participate in collaborative problem-solving situations.
>
> *Mathematics in the New Zealand Curriculum,* page 7

Students will have various ways of solving problems or presenting the process they have used and the solution. You should acknowledge successful ways of solving questions or problems, and where more effective or efficient processes can be used, encourage the students to consider other ways of solving a particular problem.

Figure It Out

YEARS 7–8

Answers Algebra: Book Two

Page 1 Which DJ?

ACTIVITY

1.

DJ	Set-up fee $	1 hour $	2 hours $	3 hours $	4 hours $	5 hours $	6 hours $
DD	40	75	110	145	195	245	295
EE	50	90	130	170	210	250	290
SS	nil	50	100	150	200	250	300

2. a. Swinging Sally

b. 3 to 5 hrs. For these hours, he is $5 cheaper than Swinging Sally.

c. 6 hrs

3. a. Dynamic Dave: 3 hrs for $127.50
Electric Eric: 2 hrs for $110. (He would charge $150 for 3 hrs.)
Swinging Sally: 3 hrs for $125

b. Swinging Sally. (She will charge the least, $125, for the longest time, 3 hrs.)

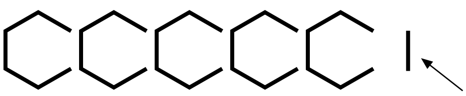
Pages 2–3 Straw Chains

ACTIVITY

1. a. 4 x 5 + 6

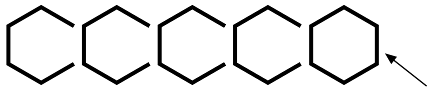
b. Possible explanations include:

Simon uses 5 straw pieces to make each hexagon. The sixth piece to complete each hexagon is also used for the first piece in the next hexagon. So the 5 hexagons require 5 x 5 = 25 straw pieces. An additional piece is needed to complete the last hexagon. The chain with 5 joined hexagons has 5 x 5 + 1 straw pieces.



5 sets of 5 straw pieces 1 extra piece

Gemma uses 5 straw pieces to make the first 4 hexagons. So the first 4 hexagons use 4 x 5 pieces. Then she uses 6 pieces to make the last hexagon. The chain with 5 joined hexagons has 4 x 5 + 6 straw pieces.

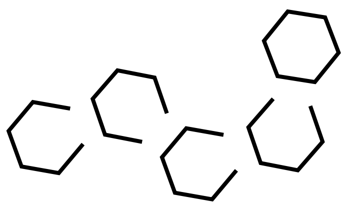


4 sets of 5 straw pieces The last hexagon has 6 pieces.

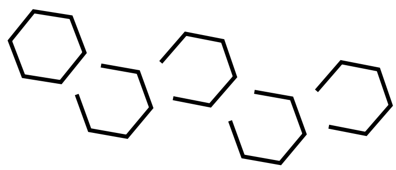
c.

Number of hexagons	Number of straw pieces	
	Simon's rule	Gemma's rule
5	5 x 5 + 1 = 26	4 x 5 + 6 = 26
10	10 x 5 + 1 = 51	9 x 5 + 6 = 51
37	37 x 5 + 1 = 186	36 x 5 + 6 = 186
96	96 x 5 + 1 = 481	95 x 5 + 6 = 481
150	150 x 5 + 1 = 751	149 x 5 + 6 = 751
497	497 x 5 + 1 = 2 486	496 x 5 + 6 = 2 486

2. a. One short cut is 4 x 5 + 6 = 26. Although the straw pieces are connected in a different way, the rule from question **1** still applies.



Another short cut is 6 + 4 x 5, with 6 pieces for the first hexagon and 5 for each of the other hexagons.

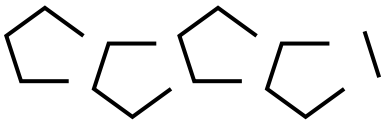


b.

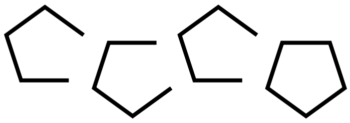
Number of hexagons	Number of straw pieces
6	31
9	46
12	61
15	76
20	101
73	366

(The easiest way to find the number of hexagons is to take off the number of straw pieces and divide the answer by 5. For example, (31 – 1) ÷ 5 = 6.)

3\. a. One short cut is 100 x 4 + 1. This is based on the short cut 4 x 4 + 1 for 4 joined pentagons and 5 x 4 + 1 for 5 joined pentagons.



b. A second short cut is 99 x 4 + 5. This is based on the short cut 3 x 4 + 5 for 4 joined pentagons and 4 x 4 + 5 for 5 joined pentagons.

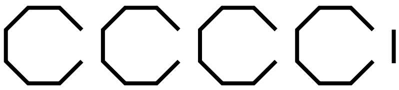


c.

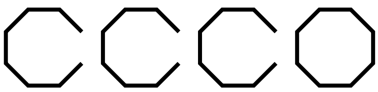
Number of pentagons	Number of straw pieces
4	17 (4 x 4 + 1)
8	33 (8 x 4 + 1)
14 ((57 – 1) ÷ 4)	57
37	149 (37 x 4 + 1)
92 ((369 – 1) ÷ 4)	369
265	1 061 (265 x 4 + 1)

4\. a. One short cut is 100 x 7 + 1 = 701.
Another is 99 x 7 + 8 = 701.

b. The first short cut is based on the short cut 4 x 7 + 1 = 29 for 4 joined octagons and 5 x 7 + 1 = 36 for 5 joined octagons.



The second short cut is based on 3 x 7 + 8 = 29 for 4 joined octagons and 4 x 7 + 8 = 36 for 5 joined octagons.



Pages 4–5 Number Crunching

ACTIVITY

1\.

Input	Output
3	21
4	27
6	39
0	3
7	45

2\.

Input	Output
4	14
5	18
9	34
0	–2
8	30

3\.

Input	Output
2	8
4	12
0	4
3	10
10	24

Input = (output – 4) ÷ 2

4\. a. 4

b. 3

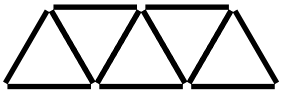
5\. a. x 4 + 1

b. x $^{-}2$ + 3

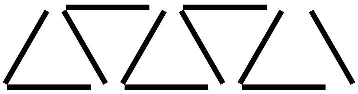
Pages 6–7 Bailey Bridges

ACTIVITY

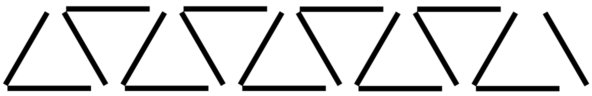
1. a.



b. A bridge with 5 triangles has 5 sets of 2 sticks and an extra stick to complete the final triangle.



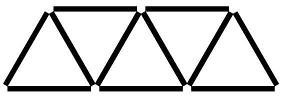
A bridge with 9 triangles has 9 sets of 2 sticks and an extra stick to complete the final triangle.



c.

Number of triangles	Number of sticks
9	9 x 2 + 1 = 19
5	5 x 2 + 1 = 11
90	90 x 2 + 1 = 181
(201 – 1) ÷ 2 = 100	201
1 563	1 563 x 2 + 1 = 3 127

2. a.



b. 6 x 2 + 3

c. A bridge with 5 triangles has 4 sets of 2 sticks and then 3 sticks for the fifth triangle. For a bridge with 7 triangles, there are 6 sets of 2 sticks and then 3 sticks for the seventh triangle.

d. 101 sticks. The short cut is 49 x 2 + 3 = 98 + 3.

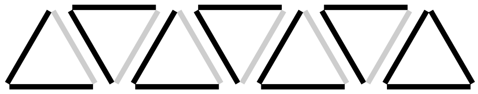
3. a. 3 + 11 x 2 = 25

b.

Number of triangles	Number of sticks	
	Rory's rule	Sali's new rule
7	6 x 2 + 3 = 15	3 + 6 x 2 = 15
12	11 x 2 + 3 = 25	3 + 11 x 2 = 25
25	24 x 2 + 3 = 51	3 + 24 x 2 = 51
100	99 x 2 + 3 = 201	3 + 99 x 2 = 201
351	350 x 2 + 3 = 703	3 + 350 x 2 = 703

c. In Rory's short cut, the last triangle has 3 sticks. In Sali's short cut, the first triangle has 3 sticks. Using either short cut will make no difference to the total.

4. a. The inclusion of 6 additional sticks helps us to see 7 triangles, each with 3 sticks.



So, altogether, there are now 7 sets of 3 or 7 x 3 sticks. But the 6 coloured sticks are not part of the bridge and so must be removed. The short cut is then 7 x 3 – 6.

b. 10 x 3 – 9 = 21

c.

Number of triangles	Number of sticks
7	7 x 3 – 6 = 15
10	10 x 3 – 9 = 21
27	27 x 3 – 26 = 55
63	63 x 3 – 62 = 127
187	187 x 3 – 186 = 375

Page 8 Table Tennis

ACTIVITY

1. a. i.

versus	**Alison**	**Barry**	**Casey**
Alison		A v B	A v C
Barry	B v A		B v C
Casey	C v A	C v B	

ii. If the shaded spaces were filled in, they would be: Alison v Alison, Barry v Barry, and Casey v Casey. This makes no sense, so the spaces are not filled.

iii. Yes. A game between Barry and Alison, B v A, is the same as a game between Alison and Barry, A v B. C v A is the same game as A v C, and C v B is the same game as B v C. So, altogether, there are just 3 possible games, B v A, C v A, and C v B. Each player will play 2 games.

b. i.

versus	Alison	Barry	Casey	Diane
Alison		A v B	A v C	A v D
Barry	B v A		B v C	B v D
Casey	C v A	C v B		C v D
Diane	D v A	D v B	D v C	

ii. 6

B v A		
C v A	C v B	
D v A	D v B	D v C

2. a.

v	A	B	C	D	E	F	G	H
A								
B	B v A							
C	C v A	C v B						
D	D v A	D v B	D v C					
E	E v A	E v B	E v C	E v D				
F	F v A	F v B	F v C	F v D	F v E			
G	G v A	G v B	G v C	G v D	G v E	G v F		
H	H v A	H v B	H v C	H v D	H v E	H v F	H v G	

Casey is not correct. The draw table for 8 players shows that there are 28 games altogether. Casey probably reasoned that doubling the number of players would double the number of games: so if there were 6 games for 4 players, there would be 2 x 6 = 12 games for 8 players.

b. One way to make predictions is to find a short cut to count the spaces in draw tables that represent the games in the tournament. So, for the tournament with 8 players, there are 8 x 8 – 8 = 56 or 8 x 7 = 56 unshaded spaces to be filled. But each game shown below the shaded spaces in the diagonal is the same as a game above the shaded spaces, so there are just (8 x 7) ÷ 2 = 28 games.

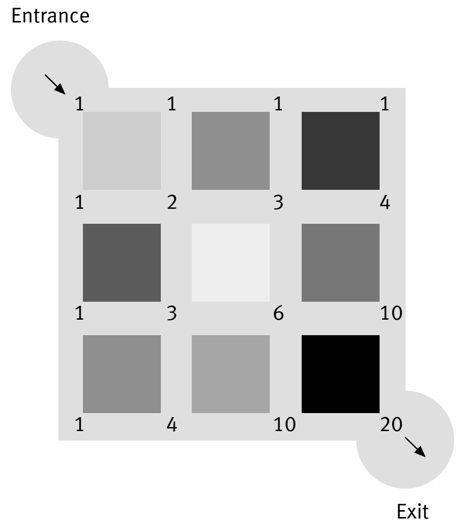
When there are 16 players, there will be (16 x 15) ÷ 2 = 120 games. Your draw table should show 120 games.

c. (100 x 99) ÷ 2 = 4 950 games

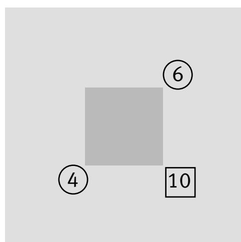
Page 9 Up the Garden Path

ACTIVITY ONE

1. a. The number of shortest routes to each intersection is shown on the map below.

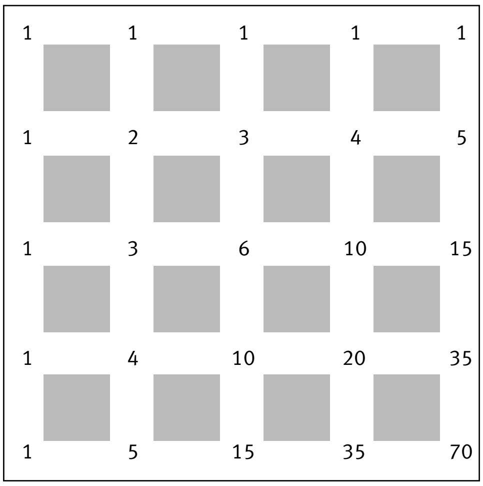


b. i. The number of shortest routes to the lower right-hand corner of each flower bed is the sum of the number of shortest routes to the adjacent corners of the same flower bed. So, for the flower bed shown below (the last bed in the middle row), there are 4 + 6 = 10 shortest routes to the lower right-hand corner of the bed.



ii. In the example given for **i**, Aisha must either pass by the upper right-hand corner or the lower left-hand corner of the last flower bed in the middle column. There are 6 ways to get to the upper right-hand corner and 4 ways to get to the lower left-hand corner. So, altogether, there are 4 + 6 = 10 ways to get to the lower right-hand corner.

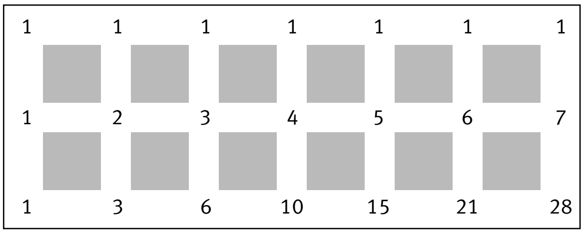
2. Aisha is correct. The pattern of numbers for the shortest routes to the intersections is shown below.



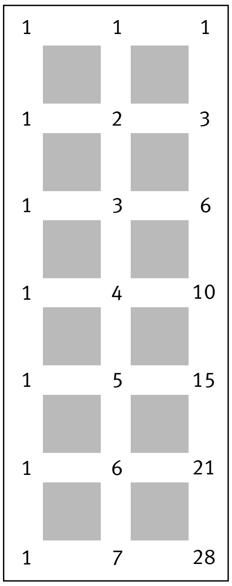
There are 70 different shortest routes from the entrance to the exit. Even if Aisha walked through the garden every day for 2 months, including weekends, the maximum number of routes needed would be 62 (2 months with 31 days).

ACTIVITY TWO

1. Two possible arrangements of the flower beds are shown below.

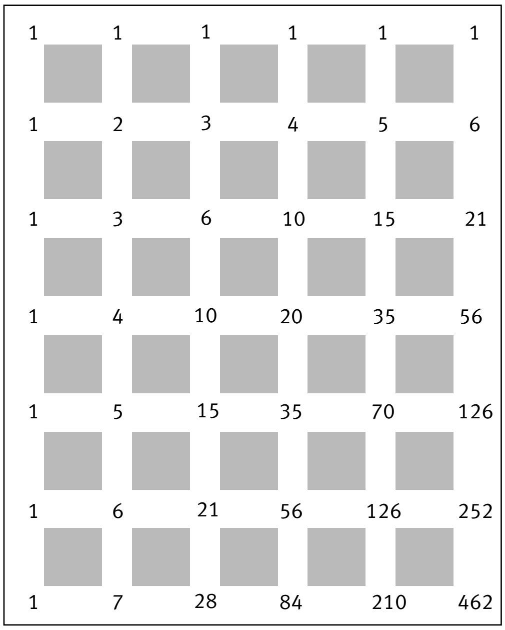


28 shortest routes through 2 rows of 6 flower beds



28 shortest routes through 6 rows of 2 flower beds

2. Aisha works on a year as 365 days (a leap year has 366 days). So she needs a minimum of 30 flower beds (6 rows of 5 flower beds or 5 rows of 6 flower beds). The number of shortest routes for this arrangement is 462.

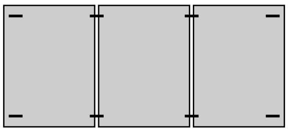


462 shortest routes through 6 rows of 5 flower beds

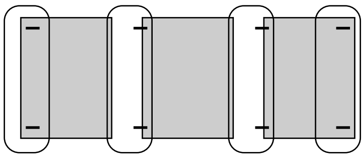
Pages 10–11 Stapled

ACTIVITY

1. a.



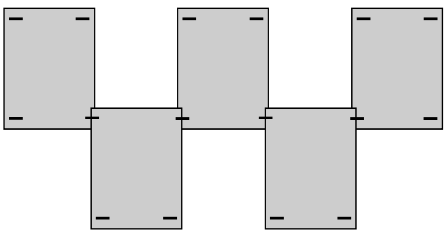
b. In the diagram for 3 pages below, each page has 2 staples. An additional 2 staples are used for the final page. So 3 pages need 3 x 2 + 2 or 4 x 2 staples. A 20-page display will need 20 x 2 + 2 or 21 x 2 = 42 staples.



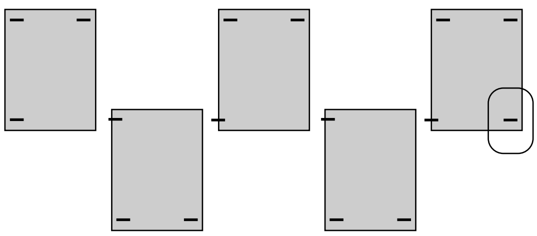
c.

Number of pages in display	Number of staples
3	4 x 2 = 8
6	7 x 2 = 14
10	11 x 2 = 22
15	16 x 2 = 32
27	28 x 2 = 56
63	64 x 2 = 128

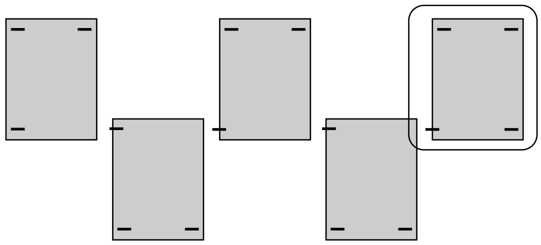
2. a.



b. Two possible short cuts are 15 x 3 + 1 = 46 staples and 14 x 3 + 4 = 46 staples. For a 5-page display, these are:



5 x 3 + 1 staples



4 x 3 + 4 staples

For the short cut 5 x 3 + 1, each of the 5 pages has 3 staples, which is 5 x 3 staples altogether. One more staple is then needed, so the 5-page display needs a total of 5 x 3 + 1 staples. So a 15-page display will need 15 x 3 + 1 = 46 staples.

For the short cut 4 x 3 + 4, each of the first 4 pages has 3 staples, which is 4 x 3 staples altogether. The last page has 4 staples, so the 5-page display needs a total of 4 x 3 + 4 staples. So a 15-page display will need 14 x 3 + 4 = 46 staples.

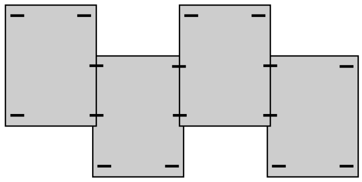
c. A 15-page display using Mr Tuwhare's pattern has 16 x 2 = 32 staples. Telea's pattern has 46 staples for 15 pages. So Telea's pattern for a 15-page display uses more staples than Mr Tuwhare's.

d.

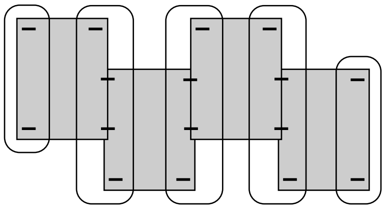
Number of pages in display	Number of staples	
	Short cut for Mr Tuwhare's pattern	Short cut for Telea's pattern
1	2 x 2 = 4	1 x 3 + 1 = 4
2	3 x 2 = 6	2 x 3 + 1 = 7
3	4 x 2 = 8	3 x 3 + 1 = 10
4	5 x 2 = 10	4 x 3 + 1 = 13
5	6 x 2 = 12	5 x 3 + 1 = 16
10	11 x 2 = 22	10 x 3 + 1 = 31
20	21 x 2 = 42	20 x 3 + 1 = 61
30	31 x 2 = 62	30 x 3 + 1 = 91
100	101 x 2 = 202	100 x 3 + 1 = 301

e. Mr Tuwhare is not entirely correct. When there is just a 1-page display, he and Telea use the same number of staples, that is, 4. However, the number of staples used for Mr Tuwhare's pattern increases by 2 for each page added, while for Telea's pattern, the number of staples used increases by 3 for each page added. So, for displays with 2 or more pages, Mr Tuwhare's pattern will always use fewer staples.

3. a.



b. The 4-poster display has
2 + 4 + 4 + 4 + 2 = 2 + 3 x 4 + 2 staples.



2 4 4 4 2

A 6-poster display will have 2 + 5 x 4 + 2 staples. So a 20-poster display will have 2 + 19 x 4 + 2 staples.

c. The short cut 2 + 19 x 4 + 2 is the same as 19 x 4 + 4. This is equivalent to 19 sets of 4 plus 1 set of 4, or 20 sets of 4. As a short cut, 20 x 4 = 80 is much simpler than 2 + 19 x 4 + 2 = 80.

d.

Number of posters in display	Number of staples	
	Mr Tuwhare's rule	Telea's rule
4	2 + 3 x 4 + 2 = 16	4 x 4 = 16
6	2 + 5 x 4 + 2 = 24	6 x 4 = 24
25	2 + 24 x 4 + 2 = 100	25 x 4 = 100
32	2 + 31 x 4 + 2 = 128	32 x 4 = 128
87	2 + 86 x 4 + 2 = 348	87 x 4 = 348

Page 12 An Artist's Delight

ACTIVITY

1. BRG, BGR, GRB, GBR, RBG, RGB

2. a. 24 different orders. You can generate four lists of equal length:

BRGO	GRBO	RBGO	ORBG
BROG	GROB	RBOG	ORGB
BGRO	GBRO	RGBO	OBRG
BGOR	GBOR	RGOB	OBGR
BORG	GORB	ROBG	OGRB
BOGR	GOBR	ROGB	OGBR

b.

Number of colours	1	2	3	4
Different orders	1	2	6	24

c. 120. (5 x 24 = 120)

3. The number of different orders is the number of colours multiplied by all the earlier numbers of colours. (For 5 colours, this is 5 x 4 x 3 x 2 x 1 = 120. For 10 colours, it would be 10 x 9 x 8 x 7 x 6 x 5 x 4 x 3 x 2 x 1 = 3 628 800.)

Page 13 Pegged Out

ACTIVITY

1. 120 m. (1 + 2 + 3 + 4 ... + 14 + 15 = 120 m. They run 240 m but only carry a peg for half that distance.)

2. 20

3. a. The calculation 7 x 8 = 56 is the total distance in metres run with and without pegs. So 28 m, which is $^1/_2$ of 56 m, is the distance run with (or without) pegs.

b. 190 m. ($^1/_2$ of 19 x 20)

4. a. A possible rule is: add 1 to the distance to the last peg, multiply this by the number of pegs, and then halve the result. (Note that the distance to the last peg is the same as the number of pegs. This could be the basis for another rule.)

b. 1 275 m. (51 x 50 ÷ 2)

Page 14 Hine's Spreadsheets

ACTIVITY

1. a. Week 0 indicates how much money ($50) Hine had before she started her regular savings in week 1.

b. The value in cell B2 is 50. So the formula =B2+4 gives 50 + 4 = 54 in cell B3.

c. B4: 58 (54 + 4)
B5: 62 (58 + 4)
B6: 66 (62 + 4)
B7: 70 (66 + 4)
B8: 74 (70 + 4)
B9: 78 (74 + 4)
B10: 82 (78 + 4)
B11: 86 (82 + 4)
B12: 90 (86 + 4)
B13: 94 (90 + 4)
B14: 98 (94 + 4)

2. a. Your spreadsheet should be similar to Grandad's spreadsheet. The values for B2–B14 are the same as those for question **1c**.

   b. The value in cell A3 is 1. So the formula =4*A3+50 gives 4 x 1 + 50 = 54. This is the same as the value calculated for cell B3 in Hine's spreadsheet.

   c. The values in the savings columns in the two spreadsheets are identical, so both formulae produce the same values.

3. a. B2: 50 (4 x 0 + 50)
   B3: 90 (4 x 10 + 50)
   B4: 98 (4 x 12 + 50)
   B5: 130 (4 x 20 + 50)
   B6: 258 (4 x 52 + 50)
   B7: 306 (4 x 64 + 50)
   B8: 450 (4 x 100 + 50)
   B9: 518 (4 x 117 + 50)

   b. Practical activity. The values are the same as in question **3a**.

   c. When Grandad's formulae are used, it is possible to immediately calculate the amount saved after any number of weeks. When the formulae in Hine's first spreadsheet are used, the amount saved can only be calculated from the amount saved in the previous week. So to find the amount saved after 1 000 weeks, the amount saved after 999 weeks must first be known, which means that Hine's first spreadsheet will use 1 000 rows of calculations to find the savings after 1 000 weeks.

   d. Hine can save $4,050. The formula used to directly calculate the amount saved after 1 000 weeks is 4 x 1 000 + 50 = $4 050. Alternatively, you could use the spreadsheet. If you enter 1000 in cell A10 (the next cell free), the value in B10 will be 4*A10+50, which is 4 x 1 000 + 50 = $4,050 (as before).

Page 15 Fish and Chips

ACTIVITY

1. a $9.60

   b. $21.00 (8 x 1.50 + 9)

   c. $t = 1.50 \times f + 1.80 \times c$

   d. Number of scoops of chips

Number of pieces of fish	0	1	2	3	4	5
0	0	1.80	3.60	5.40	7.20	9.00
1	1.50	3.30	5.10	6.90	8.70	10.50
2	3.00	4.80	6.60	8.40	10.20	12.00
3	4.50	6.30	8.10	9.90	11.70	13.50
4	6.00	7.80	9.60	11.40	13.20	15.00
5	7.50	9.30	11.10	12.90	14.70	16.50

2. a. i. $3.00

      ii. $2.00

   b. There are five possible solutions that give a total cost of $24: 8 fish and 0 scoops of chips, 6 fish and 3 scoops of chips, 4 fish and 6 scoops of chips, 2 fish and 9 scoops of chips, and 0 fish and 12 scoops of chips. A table similar to Andy's would look like the one on the following page (the $24 totals have been highlighted).

Number of scoops of chips

Number of pieces of fish

	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	$2	$4	$6	$8	$10	$12	$14	$16	$18	$20	$22	**$24**
1	$3	$5	$7	$9	$11	$13	$15	$17	$19	$21	$23	$25	$27
2	$6	$8	$10	$12	$14	$16	$18	$20	$22	**$24**	$26	$28	$30
3	$9	$11	$13	$15	$17	$19	$21	$23	$25	$27	$29	$31	$33
4	$12	$14	$16	$18	$20	$22	**$24**	$26	$28	$30	$32	$34	$36
5	$15	$17	$19	$21	$23	$25	$27	$29	$31	$33	$35	$37	$39
6	$18	$20	$22	**$24**	$26	$28	$30	$32	$34	$36	$38	$40	$42
7	$21	$23	$25	$27	$29	$31	$33	$35	$37	$39	$41	$43	$45
8	**$24**	$26	$28	$30	$32	$34	$36	$38	$40	$42	$44	$46	$48

Page 16 Island Roads

ACTIVITY

1. 6

2. a.

Number of towns	Number of roads
2	1
3	3
4	6
5	10
6	15
7	21
8	28
9	36
10	45
11	55
12	66

b. The number of roads for:

- 4 towns is the number of roads for 3 towns plus 3
- 5 towns is the number of roads for 4 towns plus 4
- 6 towns is the number of roads for 5 towns plus 5
- 7 towns is the number of roads for 6 towns plus 6

and so on.

c. 105 roads. (The table in **2a** shows that there are 66 roads for 12 towns. Continuing this pattern gives 66 + 12 = 78 roads for 13 towns, 78 + 13 = 91 roads for 14 towns, and 91 + 14 = 105 roads for 15 towns.)

Page 17 Parking Fees

ACTIVITY

1. a. $18

b. Answers should include the fact that it will cost him $20 either way; so as long as the meeting finishes on time, it makes no difference.

2. Peter's parking fees for 18 days will be $18 x 18 = $324 (not including extra fees for late Wednesdays). This is $4 more than a monthly ticket, so in any month where he will be working 18 days or more, he will be better off with a monthly ticket.

3. In a normal 5-day week when he does not have a monthly ticket, Peter spends $92 (5 x 18 + 2) on parking fees. $180 for 10 parks is equivalent to $18 per park. This is what Peter usually spends anyway, so there is no advantage in using the card on most days. But if Peter used the card on Wednesdays, when he stays later, he could save $2 each time.

4. a. $42. (92 – 50)

b. $40. (90 – 50)

Pages 18–19 The Power of 2

ACTIVITY ONE

1. a. ii. 2 units^2

   iii. 4 units^2

   iv. 8 units^2

   v. 16 units^2

   b. 32 units^2

2. A possible rule is: double the area of the enclosed square.

ACTIVITY TWO

1. 32. (2 x 2 x 2 x 2 x 2)
2. At least 7. ($2^6 = 64$ and $2^7 = 128$)
3. Yes. 20 cuts gives 1 048 576 pieces, which is more than one million pieces. There are several ways that he may have used a calculator. One way is to press the following sequence of keys (pressing the "equals" key 19 times): $\boxed{2}$ $\boxed{x}$ $\boxed{=}$ ... $\boxed{=}$.

   On some calculators, this is

   $\boxed{2}$ $\boxed{x}$ $\boxed{x}$ $\boxed{=}$ ... $\boxed{=}$.

ACTIVITY THREE

1. a. 7 moves

   b. It takes 3 moves to shift a stack of 2 lids and 15 moves to shift a stack of 4 lids.

2. It will take 31 moves to shift a stack of 5 lids. A rule for this is $2^5 - 1$. The table below shows how the rule works:

Lids	Moves	
2	3	$2 \times 2 - 1 = 2^2 - 1$
3	7	$2 \times 2 \times 2 - 1 = 2^3 - 1$
4	15	$2 \times 2 \times 2 \times 2 - 1 = 2^4 - 1$
5	31	$2 \times 2 \times 2 \times 2 \times 2 - 1 = 2^5 - 1$

Page 20 Skateboardathon

ACTIVITY

1. a.

Hours	Amount in \$	Running total in \$
1	1	1
2	3	4
3	5	9
4	7	16
5	9	25
6	11	36
7	13	49
8	15	64
9	17	81
10	19	100

   b. \$100

   c. Each number in the running total column is the square of the number in the hours column.

   d. A possible rule is: the amount raised by each boy is the square of the number of hours skated.

   e. \$576. (24 x 24)

   f. 30 hours

2. a.

Hours	Amount in \$	Running total in \$
1	2	2
2	4	6
3	6	12
4	8	20
5	10	30
6	12	42
7	14	56
8	16	72
9	18	90
10	20	110

b. One possible rule is: the amount raised by each boy is the square of the number of hours skated plus the number of hours skated. So for 48 hrs' skateboarding, the amount earned is 48 x 48 + 48 = $2,352.

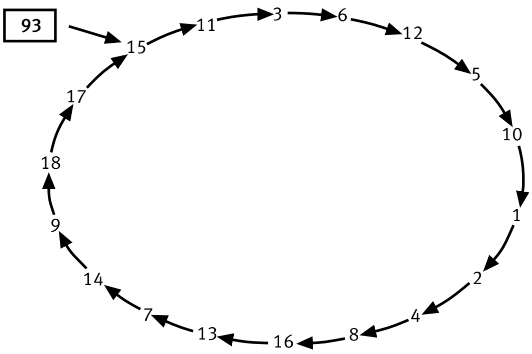
A second possible rule is: the amount raised by each boy is the product of the number of hours of skateboarding and 1 more than the number of hours of skateboarding. So for 48 hrs' skateboarding, the amount earned is 48 x 49 = $2,352.

c. 50 hrs. (50 x 51 = 2 550)

Page 21 Digit Chains

ACTIVITY

1. a. The digit rule is: double the ones digit and add the tens digit to it. The numbers after 5 in Mika's chain are: 10, 1, 2, 4, 8, 16, 13, 7, 14, 9, 18, 17, 15, 11, 3, 6, and 12. These numbers form a loop.



b. Up to 190, the multiples of 19 reduce to 19. After that, they reduce to a multiple of 19 and eventually reduce to 19 itself. For example, 133 $\longrightarrow$ 19 and 209 $\longrightarrow$ 38 $\longrightarrow$ 19.

2. a. Answers will vary. For example, for 43, the chain becomes: 43, 15, 8, 8, …

b. 2-digit numbers reduce to single-digit numbers from 1 to 9.

c. 3-digit numbers reduce to 2-digit numbers that reduce to single-digit numbers from 1 to 9.

4-digit numbers also eventually reduce to single-digit numbers from 1 to 9.

3. a. The starters that Zara has used reduce to numbers in the loop. For example, 83 becomes 25, which is in the loop. 25 becomes 2 x 5 + 5 x 5 = 35, then 3 x 5 + 5 x 5 = 40, then 20, then 10, and then finally 5.

b. Zara concludes that all numbers enter the loop. However, numbers that are multiples of 3, that is, 3, 6, 9, 12, …, don't enter the loop. Multiples of 3 that are also multiples of 9, that is, 9, 18, 27, 36, … , all become 45, which itself becomes 45. The other multiples of 3 enter a loop that has just two numbers, 30 and 15.

4. Answers will vary.

Pages 22–23 Number Puzzles

ACTIVITY

1. a. i. Subtracting 12 from the answer always gives the starting number.

Start with *n*	*n*
Add 22	*n* + 22
Subtract 6	*n* + 16
Multiply by 2	2 x *n* + 32
Subtract 8	2 x *n* + 24
Halve the answer	*n* + 12

ii. Subtracting 1 from the answer always gives the starting number.

Start with *n*	*n*
Multiply by 10	10 x *n*
Subtract 8	10 x *n* – 8
Add 16	10 x *n* + 8
Add 2	10 x *n* + 10
Divide the answer by 10	*n* + 1

iii. Subtracting 2 from the answer always gives the starting number.

Start with *n*	*n*
Add 100	*n* + 100
Multiply by 3	3 x *n* + 300
Subtract 294	3 x *n* + 6
Find a third of the answer	*n* + 2

b. Answers will vary.

2. a. The starting number is 15.

The answer is 36	36
Multiply by 2	72
Subtract 12	60
Divide by 4	15 (starting number)

b. The starting number is 10.

The answer is 36	36
Subtract 6	30
Multiply by 2	60
Add 40	100
Divide by 10	10 (starting number)

c. The starting number is 0.

The answer is 36	36
Subtract 116	−80
Divide by 8	−10
Add 10	0
Multiply by 4	0 (starting number)

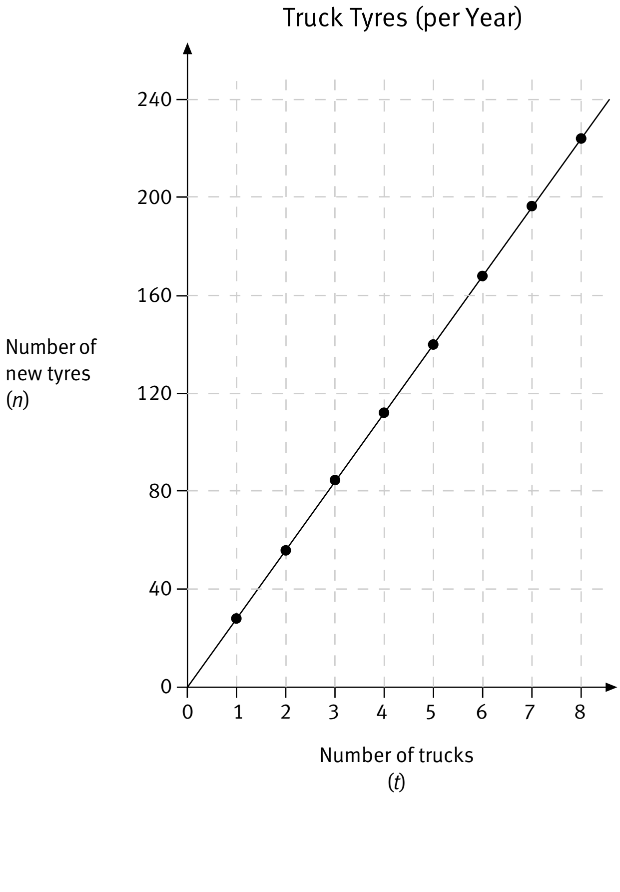
3. Answers will vary.

Page 24 Changing Tyres

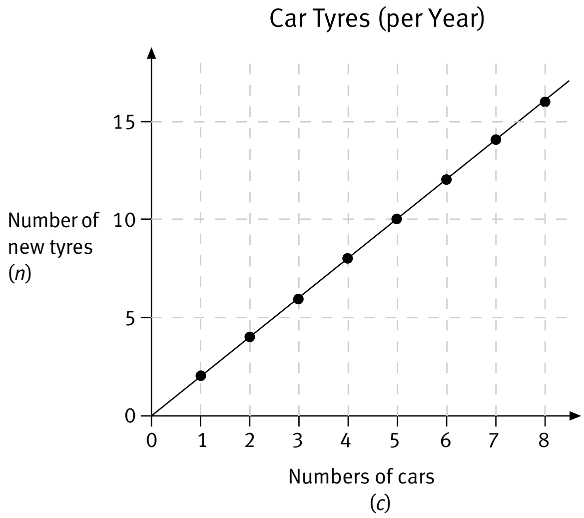
ACTIVITY

1. Every 6 months
2. a. 28 new tyres per truck

   b. $n = 28 \times t$, which is usually written as $n = 28t$
3. 6 trucks need 168 new tyres each year, and 8 trucks need 224 new tyres each year.
4.



5. a. The car tyres last for 50 000 km. On average, Mik's dad's cars travel 20 000 km each year. So, on average, the car tyres last 2.5 years. Each year, Mik's dad should budget for $5 \div 2.5 = 2$ tyres per car. If the number of tyres to be budgeted for is n and the number of cars owned is c, a formula that could be used to budget yearly for car tyres is $n = 2 \times c$ or $n = 2c$.



   b. 16

Teachers' Notes

Overview

Algebra: Book Two (level 4)

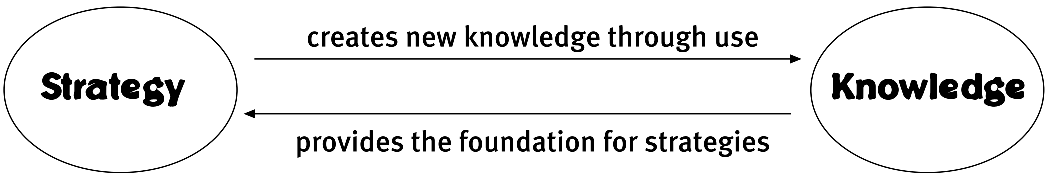
Introduction to Algebra

The teaching and learning of algebra has always posed difficulties for teachers and their students. Even today, there is no consensus about when it should be introduced and exactly what should be included in an introduction to algebra. Historically, algebra has formed an important part of the secondary curriculum. However, its inclusion as a strand in the national curriculum statement, *Mathematics in the New Zealand Curriculum*, has meant that teachers at all levels have been grappling with what should be taught at the levels at which they teach. Internationally, there is a growing consensus that the ideas of algebra have a place at every level of the mathematics curriculum.

One view is that algebra is an extension of arithmetic. Another view is that it is a completion of arithmetic. Some argue that algebra begins when a set of symbols is chosen to stand for an object or situation. Others argue that all basic operations are algebraic in nature: for example, the underlying structure of a part-whole mental strategy for adding 47 to 36 is algebraic because it may involve seeing 47 + 36 as 47 + 33 + 3, giving 50 + 30 + 3 and then 83, and such mental action constitutes algebraic thinking, in spite of the absence of algebraic-looking symbols.

The Figure It Out series aims to reflect the trends in modern mathematics education. So this series promotes the notion of algebraic thinking in which students attend to the underlying structure and relationships in a range of mathematical activities. While the student material includes only limited use of algebraic symbols, the teachers' notes show how mathematical ideas formulated in words by learners can be transformed into symbolic form. Teachers are encouraged to introduce the use of symbols in cases where they themselves feel comfortable and where they think that their students are likely to benefit.

The basis for the material in the students' books is consistent with the basis of the Number Framework, which highlights the connections between strategies students use to explore new situations and the knowledge they acquire.



To help students develop sensible strategies or short cuts for working with new mathematical situations, the activities encourage them to create their own visual and pictorial images to represent mathematical ideas and relationships. These strategies can be applied to a range of similar, as well as new, mathematical situations.

There are four Algebra books in this series for year 7–8 students:
Link (Book One)
Level 4 (Book Two)
Level 4 (Book Three)
Level 4+ (Book Four)

Which DJ?

Achievement Objectives

- devise and use problem-solving strategies to explore situations mathematically (Mathematical Processes, problem solving, level 4)
- interpret information and results in context (Mathematical Processes, developing logic and reasoning, level 4)

ACTIVITY

In this activity, students use hire-charge quotes for three DJs to complete a table that lists data for hire charges up to 6 hours. They use the data to decide which DJ has the cheapest hire charge for different time periods.

In question **3**, the students investigate the effect of having a DJ hire budget of $130, with each DJ agreeing to halve the charge for their final hour. One approach to this question is to consider the boxes in the table in which the cost is less than $130 (including the set-up fee) and then to calculate whether an additional hour, at half the normal rate, would still cost $130 or less. For example, Dynamic Dave charges $110 for 2 hours. He would normally charge $35 for the third hour but under the agreement would charge $17.50. This means that his total fee for 3 hours would be $110 + $17.50 = $127.50 (which is under $130 as required). Electric Eric charges $130 for 2 hours, so it is obvious that they won't get 3 hours out of him for $130 or less. Halving the cost of his second hour is $90 + $20 = $110. For Swinging Sally, the calculation is $100 + $25 = $125 for 3 hours. For students who have difficulty setting out the results of their analysis in a useful way, the following sequence of tables may be helpful.

If the final hour is the first hour:

DJ	Set-up fee ($)	Charge for 1 hour ($)
DD	40	40 + $^1/_2$ of 35 = 57.50
EE	50	50 + $^1/_2$ of 40 = 70
SS	0	0 + $^1/_2$ of 50 = 25

If the final hour is the second hour:

DJ	Set-up fee ($)	Charge for 1 hour ($)	Charge for 2 hours ($)
DD	40	75	75 + $^1/_2$ of 35 = 92.50
EE	50	90	90 + $^1/_2$ of 40 = 110
SS	0	50	50 + $^1/_2$ of 50 = 75

If the final hour is the third hour:

DJ	Set-up fee ($)	Charge for 1 hour ($)	Charge for 2 hours ($)	Charge for 3 hours ($)
DD	40	75	110	110 + $^1/_2$ of 35 = 127.50
EE	50	90	130	130 + $^1/_2$ of 40 = 150
SS	0	50	100	100 + $^1/_2$ of 50 = 125

Two of the DJs keep within the budget for 3 hours. All would exceed the budget for 4 hours or more. Swinging Sally is the cheaper of the DJs whose charges don't exceed $130. So Swinging Sally offers the best value in terms of the budget.

Pages 2–3 Straw Chains

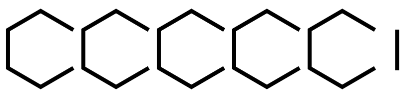
Achievement Objectives

- state the general rule for a set of similar practical problems (Algebra, level 3)
- find a rule to describe any member of a number sequence and express it in words (Algebra, level 4)
- use a rule to make predictions (Algebra, level 4)
- find and justify a word formula which represents a given practical situation (Algebra, level 4)
- use words and symbols to describe and generalise patterns (Mathematical Processes, developing logic and reasoning, level 4)
- use their own language, and mathematical language and diagrams, to explain mathematical ideas (Mathematical Processes, communicating mathematical ideas, level 4)

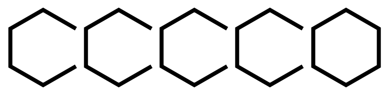
ACTIVITY

This activity asks students to analyse a pattern by breaking it into manageable units so that they can deal with it mathematically.

In question **1**, each arrangement of straw pieces is represented by a numerical short cut that can be used to count the number of pieces in the particular chain. For a chain with 100 hexagons, the number of pieces needed is 100 x 5 + 1 = 501, based on Simon's short cut, and 99 x 5 + 6 = 501, based on Gemma's short cut.



Simon's arrangement for a 5-hexagon chain



Gemma's arrangement for a 5-hexagon chain

The students should try to explain or write, in their own words, rules based on these short cuts. For Simon's arrangement, the rule is: number of straw pieces = number of hexagons x 5 + 1. The rule for Gemma's arrangement is: number of straw pieces = (number of hexagons – 1) x 5 + 6.

These rules can be expressed using algebraic symbols. So, for a chain with x hexagons, the number of pieces, y, is: $y = 5 \times x + 1$ (for Simon's rule)
and $y = 5 \times (x - 1) + 6$ (for Gemma's rule).

$5 \times x$ is usually written as $5x$, so these rules can be written as $y = 5x + 1$ (Simon's rule) and $y = 5(x - 1) + 6$ (Gemma's rule).

Although the rules are expressed differently, they are nevertheless equivalent. They produce the same value for y (the number of straw pieces) for particular values of x (the number of hexagons). The students will see this equivalence when they complete the table for question **1c**. Algebraically, the two rules are equivalent because $y = 5(x - 1) + 6$ simplifies to $y = 5 \times x - 5 \times 1 + 6$, which is equivalent to $y = 5x + 1$.

While Gemma's chain design in question **2** looks different from the earlier designs, the rules for working out the number of pieces needed for such a design with any number of hexagons are the *same* as the rules for the chains for the earlier designs. That said, the rearrangement of the hexagons may encourage some students to look for new short cuts. One such might be 6 x 5 – 4. In this case, the explanation might be as follows: There are 5 hexagons each requiring 6 pieces, but there are 4 pieces that are counted twice by this method (the pieces that are shared by two adjacent hexagons). So the correct calculation is 6 x 5 – 4.

In general, for an arrangement of this type with x joined hexagons, the number of pieces, y, is $y = 6 \times x - (x - 1)$ or $y = 6x - (x - 1)$. It can be shown that $6x - (x - 1)$ is equivalent to $5x + 1$ as per the previous two rules.

In questions **3** and **4**, the pentagon and octagon chains have rules for the number of straw pieces that are similar to those for hexagon chains. The following table shows the similarities and differences:

Rule for number of straw pieces for hexagon chains	Rule for number of straw pieces for pentagon chains	Rule for number of straw pieces for octagon chains
Number of hexagons x 5 + 1	Number of pentagons x 4 + 1	Number of octagons x 7 + 1
(Number of hexagons – 1) x 5 + 6	(Number of pentagons – 1) x 4 + 5	(Number of octagons – 1) x 7 + 8

The rules using algebraic symbols are shown below. In each case, y is the number of pieces for chains with x hexagons, x pentagons, or x octagons.

Rule for number of straw pieces for hexagon chains	Rule for number of straw pieces for pentagon chains	Rule for number of straw pieces for octagon chains
$y = 5x + 1$	$y = 4x + 1$	$y = 7x + 1$
$y = 5(x - 1) + 6$	$y = 4(x - 1) + 5$	$y = 7(x - 1) + 8$

These rules themselves form a pattern that we can use to work out the number of pieces for chains made using any polygon. So, for a chain made with 100-sided polygons, one possible rule is $y = 99x + 1$. Another is $y = 99(x - 1) + 100$. Such a chain with, for example, 1 000 polygons, will have 99 x 1 000 + 1 = 99 001 straw pieces (using the first rule).

Although working out data for a 100-sided polygon chain is unlikely to be useful, it is important that the students recognise that the rules presented could be extended in this way if required. The more *general* a rule is in mathematics, that is, the more situations it covers, the more fundamental or important it is considered to be.

Pages 4–5 Number Crunching

Achievement Objectives

- find a rule to describe any member of a number sequence and express it in words (Algebra, level 4)
- use a rule to make predictions (Algebra, level 4)
- solve simple linear equations such as $2\square + 4 = 16$ (Algebra, level 4)
- devise and use problem-solving strategies to explore situations mathematically (Mathematical Processes, problem solving, level 4)

ACTIVITY

In this activity, students investigate “number machines” that produce *output* numbers from given *input* numbers.

In questions **1** and **2**, the students use the machine operations to calculate output numbers. So, for example, the machine settings x $\boxed{4}$ + $\boxed{3}$ make the input number 6 into 6 x 4 + 3 = 27. The machine settings x $\boxed{6}$ + $\boxed{^-2}$ make the input number 5 into 5 x 6 + $^-2$ = 30 + $^-2$, which is 28. (Note that adding $^-2$ is the same as subtracting 2.)

In question **3**, the students work out the input numbers starting with the output numbers.

$\boxed{?} \longrightarrow \boxed{\text{x} \boxed{2} + \boxed{4}} \longrightarrow \boxed{12}$

To do, this they must put the machine “into reverse”. Notice how + 4 changes to – 4 and x 2 changes to ÷ 2.

$\boxed{?} \longleftarrow \boxed{\div \boxed{2} - \boxed{4}} \longleftarrow \boxed{12}$

The machine reversal subtracts 4 from 12, giving 8, and then divides 8 by 2 to give 4. So the original input number is 4. The students should convince themselves that this result is correct by running the machine forwards and checking that an input of 4 gives the desired output of 12.

In question **4a**, the students have to work out the second machine number, which has been covered.

$\boxed{3} \longrightarrow \boxed{\times \boxed{5} + \boxed{❐}} \longrightarrow \boxed{19}$

This is the same as working out the missing value in the equation 15 + ❐ = 19, which gives ❐ = 4. Using any of the other rows in the table to form equations will give the same result. Some students may realise that solving this equation is particularly easy when the input is 0. In this case, we have 0 x 5 + ❐ = 4, or ❐ = 4. In other words, the + setting of the machine is the same as the output for 0 for the machine.

This technique can be used whenever the + setting of the machine is unknown but the output for 0 is given.

In question **4b**, the students have to work out the first machine number, which has been covered.

$\boxed{4} \longrightarrow \boxed{\times \boxed{❐} + \boxed{3}} \longrightarrow \boxed{15}$

One way to do this is to put the machine into reverse.

$\boxed{4} \longleftarrow \boxed{\div \boxed{❐} - \boxed{3}} \longleftarrow \boxed{15}$

This reverse machine first works out 15 – 3 = 12 and then divides 12 by the covered machine number to get 4. So 12 ÷ ❐ = 4, giving ❐ = 3.

In question **5**, both machine numbers have been covered. In question **5a**, the machine can be represented as

$\boxed{4} \longrightarrow \boxed{\times \boxed{❐} + \boxed{❐}} \longrightarrow \boxed{17}$

The students can try different machine numbers to see which ones work. For their machine numbers to be the correct numbers, they must also work for the rest of the data in the table. For example, the students might choose x $\boxed{3}$ and + $\boxed{5}$, which work for input 4 but which won’t work for the other input data listed in the table.

Students who use the technique mentioned earlier, where the + setting for the machine equals the output for 0, will see that this setting is 1. By solving an equation such as ❐ x 4 + 1 = 17 (from the first row in the table), they will then be able to calculate that the times setting is 4. Other students may use a trial-and-improvement problem-solving strategy to achieve the same results. In either case, they should find that the machine numbers that have been covered are x $\boxed{4}$ + $\boxed{1}$.

In question **5b**, the students can once again use the “output for 0 = + setting” technique (in this case, 3) or the trial-and-improvement problem-solving strategy to find the machine settings. The only difference from the previous example is that, this time, the first machine number is negative. So they must multiply whole number input numbers by a negative number before they add another whole number. For example, when 2 is the input number for this machine, –1 is the input, and the setting is 3 (from “output for 0 = + setting”). So what negative number is the input (2) multiplied by to get a number that becomes –1 when + $\boxed{3}$ is added to it? –4 + 3 = –1, and 2 x –2 = –4, so –2 is the number we are looking for. When the students test – $\boxed{2}$ + $\boxed{3}$ with the other input and output numbers, they will find that they work in each case.

Pages 6–7

Bailey Bridges

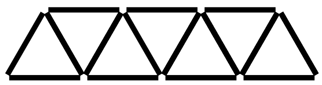
Achievement Objectives

- find and justify a word formula which represents a given practical situation (Algebra, level 4)
- find a rule to describe any member of a number sequence and express it in words (Algebra, level 4)
- use a rule to make predictions (Algebra, level 4)
- devise and use problem-solving strategies to explore situations mathematically (Mathematical Processes, problem solving, level 4)
- use words and symbols to describe and generalise patterns (Mathematical Processes, developing logic and reasoning, level 4)

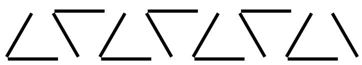
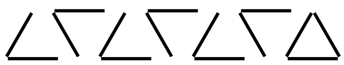
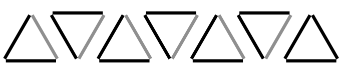
ACTIVITY

In this activity, students work with four different short cuts for counting the sticks needed to make Bailey bridges. The short cuts arise from different ways of visualising the structure of Bailey bridges. The students explain the short cuts and use them to predict the number of sticks in Bailey bridges with *any* number of triangles.

While some students will be able to relate the short cuts directly to the bridge diagrams, others may need to use sticks to build the models for themselves to help them make the connections between the models and the short cuts.

As an example, this Bailey bridge has 7 triangles: 

The structure of the bridge may be visualised in several ways. The four different ways explored in the students' activities are shown in the table below.

Arrangement of sticks	Short cut expression for number of sticks in 7 triangles	Rule in words for number of sticks in *any* number of triangles	Algebraic rule for number of sticks, y, for x triangles
	7 x 2 + 1 = 15	number of triangles x 2, plus 1	$y = 2x + 1$
	6 x 2 + 3 = 15	(1 fewer than the number of triangles) x 2, plus 3	$y = 2(x - 1) + 3$
	3 + 6 x 2 = 15	3, plus (1 fewer than the number of triangles) x 2	$y = 3 + 2(x - 1)$
	7 x 3 – 6 = 15	number of triangles x 3, minus 1 fewer than the number of triangles	$y = 3x - (x - 1)$

Note, for example, how the short cut 7 x 2 + 1 generalises to the algebraic rule $y = x \times 2 + 1$ or more simply, $y = 2x + 1$. An important point is that, while the rules may be different, the outcome is the same. A bridge that has 7 triangles uses 15 sticks, no matter which rule is used. In questions **1–4**, the students explain how each short cut works. These explanations are, in effect, the rules or generalisations arising from the short cuts. The students may need support and time to work with other students in refining their rule explanations, which are the key to this work in algebraic thinking.

While it is not intended that students at this level work with algebraic rules expressed in symbolic form, some students may benefit from such activity. It is likely, however, that even the most able students will only benefit in situations where their teachers are confident in their own understanding of algebra.

The simplest algebraic rule for the number of sticks in Bailey bridges is $y = 2x + 1$. The other rules all reduce to this rule. So $y = 2(x - 1) + 3$ becomes $y = 2 \times x - 2 \times 1 + 3$, which further reduces to $y = 2x - 2 + 3$ and then to $y = 2x + 1$. The algebraic manipulations needed to produce $y = 2x + 1$ from the other two rules are:

$$\begin{aligned} y &= 3 + 2(x - 1) \\ &= 3 + 2 \times x - 2 \times 1 \\ &= 3 + 2x - 2 \\ &= 1 + 2x \\ &= 2x + 1 \end{aligned}$$

$$\begin{aligned} y &= 3x - (x - 1) \\ &= 3x - x + 1 \\ &= 2x + 1 \end{aligned}$$

Note how the second subtraction sign in $y = 3x - (x - 1)$ becomes an addition sign (because a negative minus a negative is a positive), so that $y = 3x - x + 1$. If it didn't change like this, the simplified form would be $y = 2x - 1$, which is not correct.

Page 8 Table Tennis

Achievement Objectives

- devise and use problem-solving strategies to explore situations mathematically (Mathematical Processes, problem solving, level 4)
- use a rule to make predictions (Algebra, level 4)
- interpret information and results in context (Mathematical Processes, developing logic and reasoning, level 4)
- find and justify a word formula which represents a given practical situation (Algebra, level 4)
- use their own language, and mathematical language and diagrams, to explain mathematical ideas (Mathematical Processes, communicating mathematical ideas, level 4)

ACTIVITY

This activity provides visual support for students to predict the number of games in various draws. After completing draw tables for question **1**, in which the students work out the number of games for tournaments with 3 and then 4 players, they are able to make predictions for the number of games with 8, 16, and then 100 players.

In the table for 3 players, there are 3 x 3 = 9 spaces to fill (figure 1). The three diagonal spaces can be filled with A v A, B v B, and C v C. This is not sensible, so the spaces on the diagonal are shaded (figure 2). So there are now 3 x 3 – 3 = 6 spaces to fill (figure 3). A v B is the same game as B v A, A v C is the same game as C v A, and B v C is the same game as C v B. So, in practice, only one-half of the 3 x 3 – 3 = 6 spaces need to be filled (figure 4). The number of games in a tournament for 3 players is therefore $^1/_2$ of (3 x 3 – 3), which is $^1/_2$ of 6 = 3.

v	A	B	C
A			
B			
C			

Figure 1

v	A	B	C
A			
B			
C			

Figure 2

v	A	B	C
A		A v B	A v C
B	B v A		B v C
C	C v A	C v B	

Figure 3

v	A	B	C
A			
B	B v A		
C	C v A	C v B	

Figure 4

A tournament with 4 players has $^1/_2$ of (4 x 4 – 4) or $^1/_2$ of 12, which is 6 games. A tournament with 8 players has $^1/_2$ of (8 x 8 – 8), or $^1/_2$ of 56, which is 28 games. Draw tables for these are given in the Answers.
The table below shows how the pattern works.

Number of players	Pattern for number of games	Number of games
2	$^1/_2$ of (2 x 2 – 2)	1
3	$^1/_2$ of (3 x 3 – 3)	3
4	$^1/_2$ of (4 x 4 – 4)	6
8	$^1/_2$ of (8 x 8 – 8)	28
10	$^1/_2$ of (10 x 10 – 10)	45
100	$^1/_2$ of (100 x 100 – 100)	4 950

We can use algebraic symbols to show this pattern. So, for x players, there are $^1/_2$ of $(x \times x - x)$ games, or $\frac{(x^2 - x)}{2}$.

Another way to visualise the pattern is to notice, for example, that in the table for 4 players there are 4 rows of 3 spaces to be filled. The space that lies on the diagonal is excluded, so there are 4 x 3 = 12 spaces to be filled. But half of the games in these spaces are duplicates (A v B = B v A, and so on), so the number of games is $^1/_2$ of (4 x 3) or $^1/_2$ of 12, which is 6 games. The table below shows how the pattern works:

Number of players	Pattern for number of games	Number of games
2	$^1/_2$ of (2 x 1)	1
3	$^1/_2$ of (3 x 2)	3
4	$^1/_2$ of (4 x 3)	6
8	$^1/_2$ of (8 x 7)	28
10	$^1/_2$ of (10 x 9)	45
100	$^1/_2$ of (100 x 99)	4 950

We can also use algebraic symbols to show this pattern. So, for x players, there are $^1/_2$ of $x \times (x - 1)$ or $\frac{x(x - 1)}{2}$ games.

While the two algebraic rules look different, they are equivalent. The second rule, $\frac{x(x - 1)}{2}$, can be expanded to give $\frac{(x \times x - x \times 1)}{2}$, which simplifies to $\frac{(x^2 - x)}{2}$, which is the first rule.

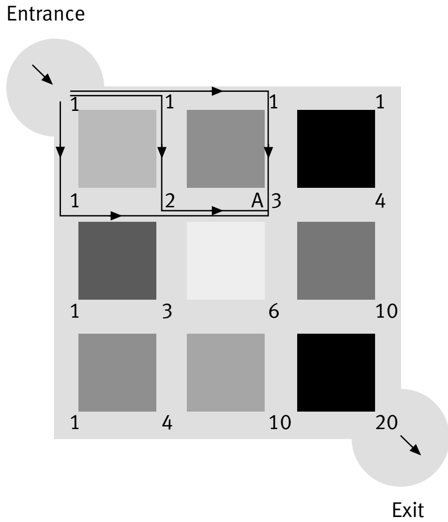
Page 9 Up the Garden Path

Achievement Objectives

- devise and use problem-solving strategies to explore situations mathematically (Mathematical Processes, problem solving, level 4)
- find a rule to describe any member of a number sequence and express it in words (Algebra, level 4)
- use a rule to make predictions (Algebra, level 4)
- use words and symbols to describe and generalise patterns (Mathematical Processes, developing logic and reasoning, level 4)
- use their own language, and mathematical language and diagrams, to explain mathematical ideas (Mathematical Processes, communicating mathematical ideas, level 4)

ACTIVITY ONE

In this activity, students systematically explore the different shortest routes through a garden designed in the shape of a grid.



This garden grid with 9 flower beds shows the number of shortest routes to each intersection as well as the 3 shortest paths to the intersection A.

The students will need a large copy of the grid so that they can trace enough of the shortest routes to be able to see number patterns on their grid.

For some students, the number patterns may be more obvious when the grid is rotated a quarter-turn to the right. Then the numbers appear as in the table below, often referred to as Pascal's triangle. The triangle of numbers can be extended indefinitely.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

Pascal's Triangle

Blaise Pascal (1623–1662) was a French mathematician who, together with another French mathematician, Pierre de Fermat (1601–1665), created much of what we know as probability theory.

The relationship within each set of 3 numbers, such as those shown below, can be used to extend Pascal's triangle to get the eighth row of numbers, the ninth row, and so on.

1	2
3	

$1 + 2 = 3$

4	6
10	

$4 + 6 = 10$

10	5
15	

$10 + 5 = 15$

The students will need to use this pattern to figure out the answers to question **2** in the activity.

ACTIVITY TWO

This activity asks students to design gardens on the basis of a given number of shortest routes. One way to design a garden with 28 shortest routes for question **1** is to arrange the flower beds in a rectangular array, that is, to reverse the thinking used in **Activity One**. In doing this, the students are effectively extending the numbers in Pascal's triangle until they find the number 28.

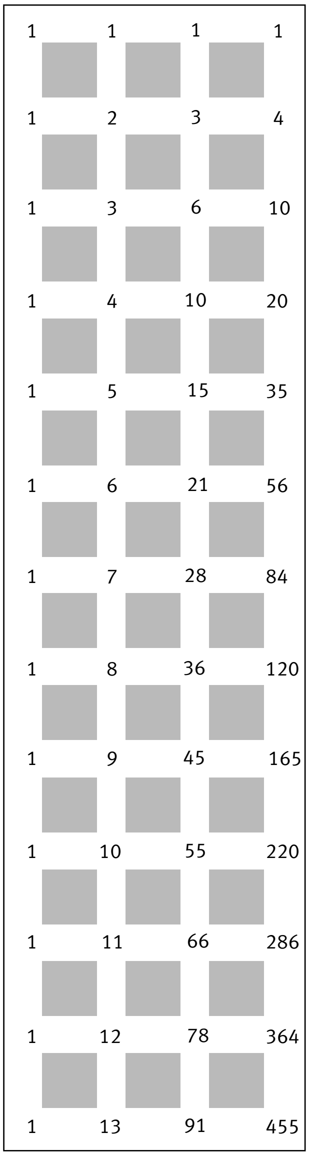
The diagrams in the Answers show two possible flower bed arrangements with exactly 28 shortest routes through the flower beds. One garden shows a 6 by 2 rectangular arrangement of flower beds and the other shows a 2 by 6 rectangular arrangement.

The only other rectangular arrays of beds that have exactly 28 shortest routes are a single row of 27 beds or a single column of 27 beds.

Aisha's garden can be extended, using Pascal's triangle, to a garden that has at least 365 shortest routes, one for each day of the year. The students will need to use square grid paper to design this garden.

The garden shown in the Answers has 6 rows of 5 or a total of 30 flower beds, which is the smallest number of flower beds that a garden with at least 365 different shortest routes can have. In fact, there are 462 shortest routes altogether. This garden could obviously also be rotated through 90 degrees to give an arrangement with 5 rows of 6 beds.

The students might explore further to see if they can find the next smallest garden that has at least 365 shortest routes. The garden below has 36 flower beds and 455 shortest routes. Unless it is a leap year, Aisha will need to choose just one more route on the last day of the year when she has completed the 364 routes to the intersection marked below. (She will have 91 options for this last route.)



There are 364 shortest routes to this intersection.

Pages 10–11 Stapled

Achievement Objectives

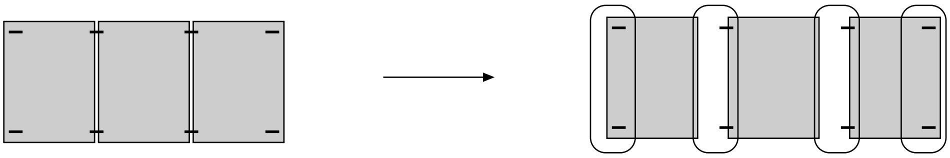
- devise and use problem-solving strategies to explore situations mathematically (Mathematical Processes, problem solving, level 4)
- find a rule to describe any member of a number sequence and express it in words (Algebra, level 4)
- use a rule to make predictions (Algebra, level 4)

ACTIVITY

This activity asks students to explore different arrangements for stapling pages and posters for a wall display.

The students could set up similar displays on their classroom walls. They should also draw each display, including the staples, to help them visualise how each short cut and associated rule works.

In question **1**, notice how the stapling works for a 3-page display:

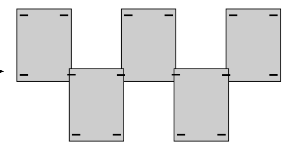


Each page has 2 staples, and the last page has an additional 2 staples. So a display with 30 pages has 30 x 2 + 2 staples. This is the same as 30 x 2 + 1 x 2 or 31 x 2. This means that a 100-page display has 101 x 2 = 202 staples. Using algebraic symbols, a display with x pages will have x x 2 + 2 or (x + 1) x 2 staples, or more simply, 2(x + 1) or 2x + 2. Not everyone is easily convinced that 2x + 2 and 2(x + 1) are equivalent, but completing a table like the one below usually helps.

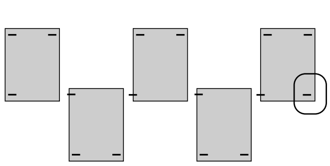
x	$2x + 2$	$2(x + 1)$
1	2 x 1 + 2 = 4	2 x (1 + 1) = 4
3	2 x 3 + 2 = 8	2 x (3 + 1) = 8
7	2 x 7 + 2 = 16	2 x (7 + 1) = 16
73	2 x 73 + 2 = 148	2 x (73 + 1) = 148

As indicated earlier, it is not generally intended for students at this level to work with algebraic rules expressed in symbolic form, but some students may benefit from doing so.

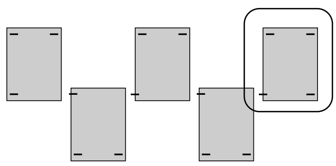
In question **2**, two possible short cuts for this 5-page display



arise from the following diagrams:



5 x 3 + 1 staples



4 x 3 + 4 staples

So, for a 100-page display, the number of staples is 100 x 3 + 1 or 99 x 3 + 4. The value for each short cut is 301. These short cuts give rise to the algebraic rules x x 3 + 1 and (x – 1) x 3 + 4 for x pages. The rules are written as 3x + 1 and 3(x – 1) + 4 respectively. The two short cuts are easily shown to be equivalent algebraically: 3 x (x – 1) + 4 = 3(x – 1) + 4 or 3x – 3 + 4, which is 3x + 1, although doubters may need to complete a table similar to the one shown for question **1**.

In questions **2d–e**, the students compare the display from question **1** (Mr Tuwhare's) with Telea's in **2a** to see which uses fewer staples. They should find that for the completed data in the table, Mr Tuwhare's pattern uses fewer staples when there is more than 1 page in the display. This claim can be extended to all displays with more than 1 page by first noticing that the number of staples increases by 2 for each page added to Mr Tuwhare's display and increases by 3 for each page added to Telea's display. So the number of staples in Telea's display increases at a faster rate than it does for Mr Tuwhare's display.

In question **3**, this 4-poster display has 2 + 3 x 4 + 2 or 3 x 4 + 4 staples. The short cut 3 x 4 + 4 is the same as 4 x 4. A 10-poster display will have 9 x 4 + 4 staples or 10 x 4 staples, and a 100-poster display will have 99 x 4 + 4 or 100 x 4 staples. A display with x posters will therefore have $4(x - 1) + 4$ or $4x$ staples.

Page 12 An Artist's Delight

Achievement Objectives

- devise and use problem-solving strategies to explore situations mathematically (Mathematical Processes, problem solving, level 4)
- find a rule to describe any member of a number sequence and express it in words (Algebra, level 4)
- use a rule to make predictions (Algebra, level 4)
- use words and symbols to describe and generalise patterns (Mathematical Processes, developing logic and reasoning, level 4)
- record information in ways that are helpful for drawing conclusions and making generalisations (Mathematical Processes, communicating mathematical ideas, level 4)

ACTIVITY

In this activity, students consider the number of ways that a set of colours can be placed in order.

It might be helpful to use coloured multilink cubes to explore the possible orders that Turi could colour her design. For example, there are 2 orders for blue and green: either blue first and then green or green first and then blue.

B G   G B

When Turi adds another colour, red, she could either use a trial-and-error strategy or a systematic strategy to figure out all the 6 possible orders to colour her design. While a trial-and-error strategy is fine when the number of colours is small, the number of ways in which the colours can be ordered increases very quickly. Encourage the students to develop a systematic strategy so that they can convince themselves and others that they have found *all* possible orders. Four such systematic strategies are shown below.

Strategy 1: Use one colour in the first, then the second, and finally the third position, and record all the possibilities for the other colours. For example, put red in the first position:

R B G   R G B

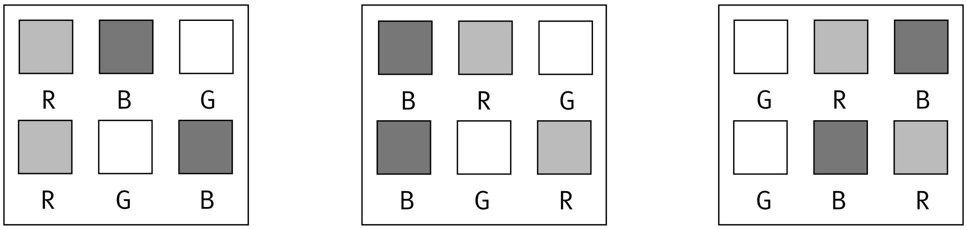
Then put red in the second position:

B R G   G R B

Finally, put red in the third position:

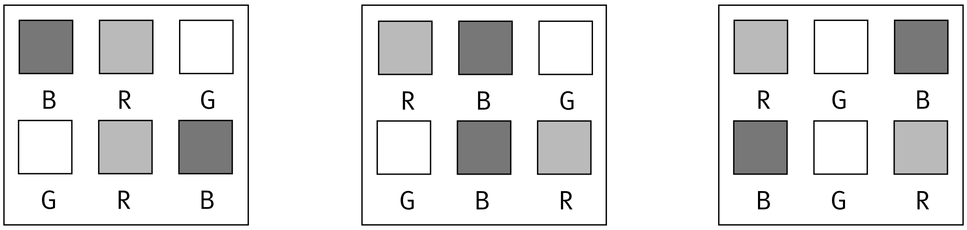
B G R   G B R

Strategy 2: Use each colour in the first position.



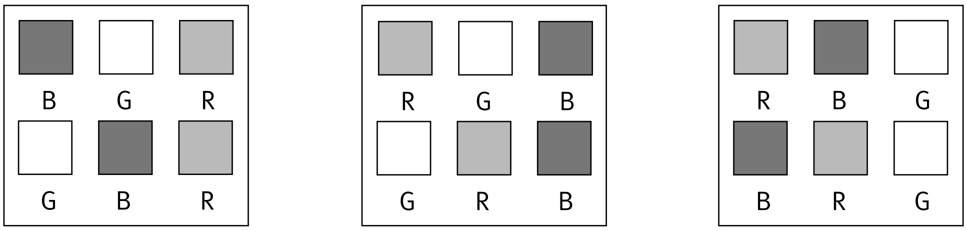
Red first Blue first Green first

Strategy 3: Use each colour in the second position.



Red second Blue second Green second

Strategy 4: Use each colour in the third position.

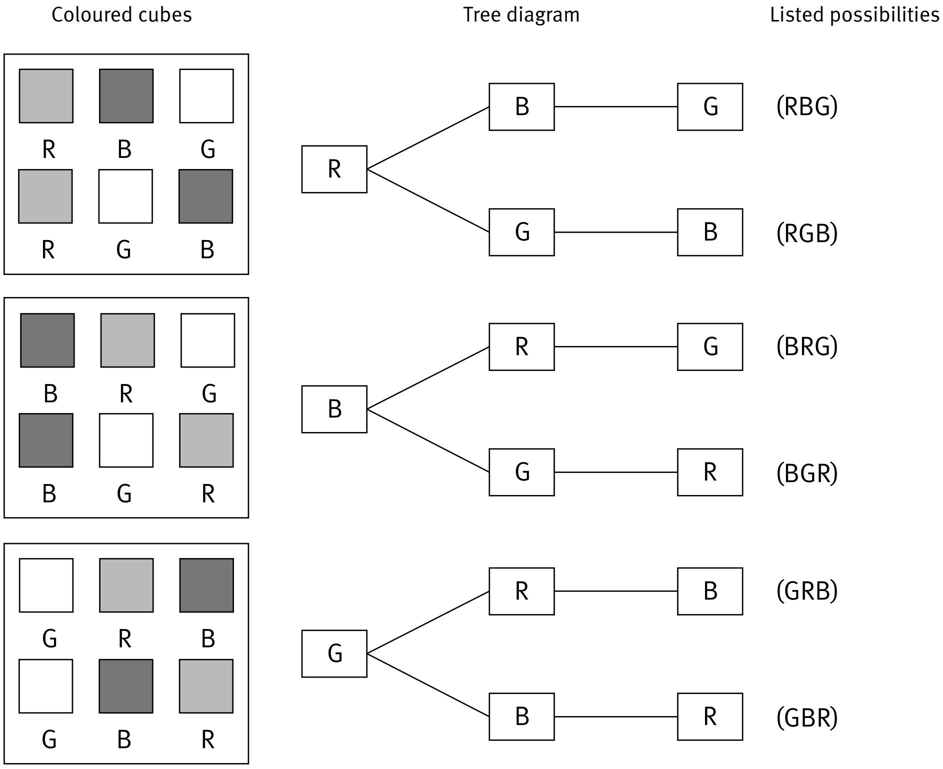


Red third Blue third Green third

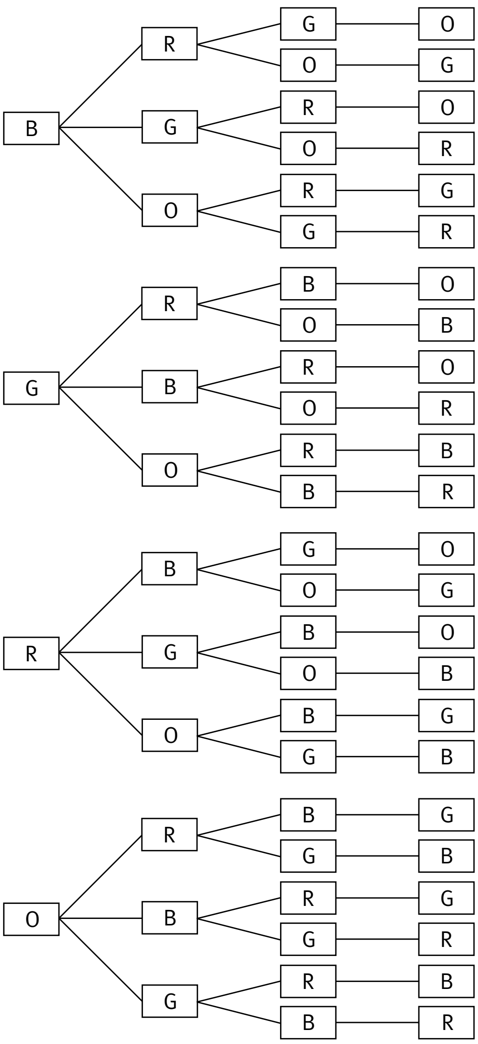
All the strategies produce the same 6 possible orders of the colours.

In question **2**, the students investigate adding a fourth colour, orange. It turns out that there are 24 orders altogether, so it will not be practicable to make them all with multilink cubes. Some students may want to try drawing the different orders on square grid paper, but it will be more profitable for the students to see if they can find a way to record their results systematically. Most students, however, are unlikely to be able to get started on their own initiative. A sensible way to record their results for 4 or more colours involves using the diagram for strategy 2, with each colour in the first position.

Instead of putting the possible arrangements for red first, blue first, and green first side by side, they can be arranged below each other to form the basis of a tree diagram.



The tree diagram shows that there are 2 orders for using 3 colours with red first. There are also 2 orders for 3 colours with blue first and 2 orders for 3 colours with green first. So altogether, there are 2 + 2 + 2 = 3 x 2 or 6 orders for 3 colours. A tree diagram for 4 colours might look like this:



A table can help the students to see the pattern at work here:

Number of colours	Number of orders to paint in	Explanation	Pattern
1	1	There is only 1 way to arrange 1 colour.	1
2	2	There are 2 ways to arrange 2 colours.	2
3	6	For each of the 3 colours, there are 2 ways to arrange the remaining colours.	3 x (2)
4	24	For each of the 4 colours, there are 3 x 2 = 6 ways to arrange the remaining colours.	4 x (3 x 2)
5	120	For each of the 5 colours, there are 4 x 3 x 2 = 24 ways to arrange the remaining colours.	5 x (4 x 3 x 2)
10	3 628 800	For each of the 10 colours, there are 9 x 8 x 7 x 6 x 5 x 4 x 3 x 2 = 362 880 ways to arrange the remaining colours.	10 x (9 x 8 x 7 x 6 x 5 x 4 x 3 x 2)

There is a shorter way to write numbers in the form 3 x 2 x 1, and so on. For 3 x 2 x 1, this is 3!, and we say "factorial" for the exclamation mark. Factorials are used a lot in probability, and they are also interesting in that they grow very quickly. The following table shows the factorial pattern:

Pattern	Factorial	Value
2 x 1	2!	2
3 x 2 x 1	3!	6
4 x 3 x 2 x 1	4!	24
10 x 9 x 8 x 7 x 6 x 5 x 4 x 3 x 2 x 1	10!	3 628 800
n x (n – 1) x (n – 2) … x 1	n!	

Note that all mathematical and scientific calculators have a factorial function key.

Page 13 Pegged Out

Achievement Objectives

- devise and use problem-solving strategies to explore situations mathematically (Mathematical Processes, problem solving, level 4)
- find a rule to describe any member of a number sequence and express it in words (Algebra, level 4)
- use a rule to make predictions (Algebra, level 4)
- use words and symbols to describe and generalise patterns (Mathematical Processes, developing logic and reasoning, level 4)

ACTIVITY

In this activity, students add consecutive numbers that represent distances of pegs from a starting position.

In question **1**, the students must first understand the nature of the relay competition. This means that they must see that for a race with 15 pegs spaced 1 metre apart, each competitor must carry pegs a total distance of 120 metres, which is found from 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 + 15.

In question **2**, they are likely to use a trial-and-improvement problem-solving strategy to work out that if each competitor were to run 210 metres carrying pegs, there must be 20 pegs involved. In question **3**, however, they follow the beginnings of Trudie's short cut to work out how far each competitor must run carrying pegs when there are just 7 pegs. Trudie's short cut is reasonably sophisticated and may require some discussion to help the students understand how it works. In question **3b**, they see if they can extend Trudie's reasoning to a race with 19 pegs.

The patterns in the table below show how the rule given in the Answers can be extended to a race with x pegs.

Number of pegs	Total distance to run	Total distance to run carrying pegs
7	7 x 8	$^1/_2$ of (7 x 8)
10	10 x 11	$^1/_2$ of (10 x 11)
15	15 x 16	$^1/_2$ of (15 x 16)
x	x x $(x + 1)$	$^1/_2$ of x x $(x + 1)$ or $^1/_2$ of $x(x + 1)$

When a race has 50 pegs, the distance each competitor must run carrying pegs is $^1/_2$ x 50 x 51, which is $^1/_2$ x 2 550, or 1 275 metres.

Some students may be interested in exploring ways to add consecutive numbers. At the age of 7, in his first year at primary school, Carl Friedrich Gauss (1777–1855), who later became one of the world's greatest mathematicians, suggested to his teachers a way to add the numbers from 1 to 100. For numbers from 1 to 10 rather than 1 to 100, his short cut is:

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 \longrightarrow \begin{array}{ccccccccc} 1 & + & 2 & + & 3 & + & 4 & + & 5 \\ 10 & + & 9 & + & 8 & + & 7 & + & 6 \\ \hline 11 & + & 11 & + & 11 & + & 11 & + & 11 \end{array}$$

The sum is then 5 x 11 = 55. For the numbers from 1 to 100, it is 50 x 101 = 5 050.

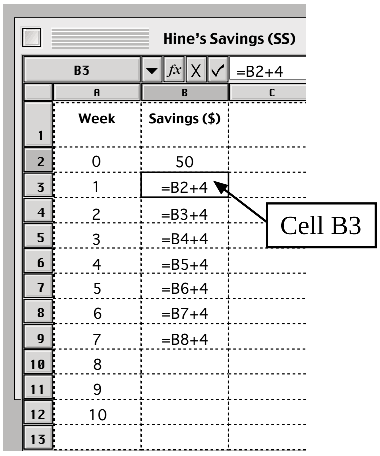
Page 14 Hine's Spreadsheets

Achievement Objectives

- find a rule to describe any member of a number sequence and express it in words (Algebra, level 4)
- use a rule to make predictions (Algebra, level 4)
- use equipment appropriately when exploring mathematical ideas (Mathematical Processes, problem solving, level 4)

ACTIVITY

In this activity, students explore spreadsheet work. The first exercises cover material that they may have considered earlier, for example, on page 14 of *Algebra: Book One* in this series of Figure It Out. In the spreadsheet on the following page, the formula =B2+4 in cell B3 indicates that the value in cell B3 is 4 more than the value in B2. The value in cell B2 is 50, so the value in cell B3 is 50 + 4 = 54. After her saving in week 1, Hine has $54 in her savings account. Note that in week 0, Hine has $50. We can interpret this as the amount she has in her account before she began her regular savings programme.



Note that the formula =B3+4 in cell B4 indicates that in week 2, Hine has $4 more in her account than in week 1. Similarly, the formula =B4+4 in cell B5 indicates that in week 3, Hine has $4 more in her account than in week 2, and so on.

In questions **2** and **3**, Grandad's spreadsheet formulae are different from those used by Hine, yet the different spreadsheets produce the same results. The following table shows the link between the two spreadsheets:

Week	Hine's spreadsheet		Grandad's spreadsheet	
	Formulae	Calculations	Formulae	Calculations
0	50	No calculation	=4*A2+50	4 x 0 + 50 = 50
1	=B2+4	50 + 4 = 54	=4*A3+50	4 x 1 + 50 = 54
2	=B3+4	54 + 4 = 58	=4*A4+50	4 x 2 + 50 = 58
3	=B4+4	58 + 4 = 62	=4*A5+50	4 x 3 + 50 = 62
10	=B11+4	Need the value for week 9	=4*A12+50	4 x 10 + 50 = 90
100	=B101+4	Need the value for week 99	=4*A102+50	4 x 100 + 50 = 450

Note that in Hine's formulae, the total saved after any period is calculated by adding $4 to the total saved in the previous week. When the total saved for the previous week is unknown, it is not possible to calculate the current total saved.

Grandad's formulae, on the other hand, directly calculate the total saved for any given week. They work by calculating how much Hine would have if she saved $4 each week for a given number of weeks and then adding this to the $50 she started with. The total saved in 100 weeks is calculated from 4 x 100 + 50 = 450, and the total saved in 456 weeks is calculated from 4 x 456 + 50 = 1 874. The total saved in x weeks is 4 x x + 50 or $4x + 50$. If we use y for the total saved after x weeks, then $y = 4x + 50$.

Notice also that repeatedly adding 4 as in Hine's formulae becomes multiplying by 4 in Grandad's formulae.

=B4+4 =4*A5+50

Adding 4 repeatedly is the same as multiplying by 4.

So, for example, 4 + 4 + 4 + 4 + 4 = 5 x 4.

You could use this relationship in extension work in which the students could look for a rule for number sequences where the increase or decrease between successive terms does not change. For example, successive terms in the sequence 3, 5, 7, 9, 11, … increase by 2.

Position in sequence	Term
1	3
2	5
3	7
4	9
5	11

As a consequence, we can reason that the rule for the pattern can be expressed algebraically as $y = 2x + ❐$. The 2 in $2x$ represents the constant difference, 2, between successive terms. The value that goes in ❐ can be worked out by applying the rule to the first term in the sequence, that is, when $x = 1$, the term is 3. So, the value that goes in the box is 1, and the rule is $y = 2x + 1$.

Position in sequence	Term
1	8
2	13
3	18
4	23
5	28

The difference between successive terms is 5, so the rule is $y = 5x + ❐$. The value for ❐ is 3 when $x = 1$ (the first term), so the rule is $y = 5x + 3$.

Position in sequence	Term
1	17
2	30
3	43
4	56
5	69

The difference between successive terms is 13, so the rule is $y = 13x + ❐$. The value for ❐ is 4 when $x = 1$ (the first term), so the rule is $y = 13x + 4$.

Students who are interested in this might like to try finding the rules for the following sequences:

Position	Term
1	5
2	8
3	11
4	14
5	17

(Rule: $3x + 2$)

Position	Term
1	3
2	7
3	11
4	15
5	19

(Rule: $4x – 1$)

Page 15 Fish and Chips

Achievement Objectives

- interpret information and results in context (Mathematical Processes, developing logic and reasoning, level 4)
- use words and symbols to describe and generalise patterns (Mathematical Processes, developing logic and reasoning, level 4)
- use a rule to make predictions (Algebra, level 4)

ACTIVITY

In this activity, students develop rules or formulae that can be used to determine the total prices for meals of fish and chips.

In question **1**, the students use the Fresh Fish Shop's display of their prices to make a "double-entry table" or ready reckoner that can be used to read off the price of a range of fish and chip orders. In question **2**, however, only the formula for the total cost is given. The students must make a ready reckoner table from this formula and search the table to locate total charges of $24. Note that the table will need to be extended to 8 fish x 12 scoops of chips to cover every possibility. Some students will discover this by trial and error, but others may reason that 24 can be divided by both 3 and 2, which gives the possible orders 8 fish x $3 = $24 and 12 chips x $2 = $24 respectively.

Some students may argue that neither of these orders are orders of fish *and* chips and that there are really only three possible orders; strictly speaking, they are correct!

The table in the Answers shows five entries representing purchases of $24. These are:

Number of pieces of fish	Number of scoops of chips
0	12
2	9
4	6
6	3
8	0

Note the pattern in this table. For every 2 additional pieces of fish, 3 fewer scoops of chips can be bought for a purchase price of $24.

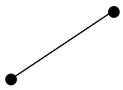
Page 16 Island Roads

Achievement Objectives

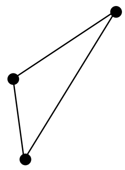
- devise and use problem-solving strategies to explore situations mathematically (Mathematical Processes, problem solving, level 4)
- find a rule to describe any member of a number sequence and express it in words (Algebra, level 4)
- use a rule to make predictions (Algebra, level 4)
- find and justify a word formula which represents a given practical situation (Algebra, level 4)

ACTIVITY

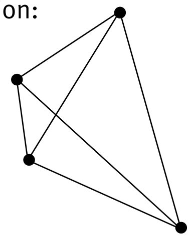
This activity asks students to systematically explore the road requirements that link 2 towns, then 3 towns, 4 towns, and so on. One way to do this is to draw a diagram for each situation:



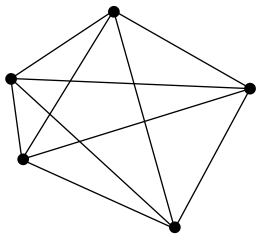
2 towns, 1 road



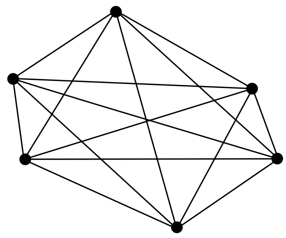
3 towns, 3 roads



4 towns, 6 roads



5 towns, 10 roads



6 towns, 15 roads

If the data from the diagrams above is put into a table, the pattern can be extended up to 12 towns and then to 15 towns.

Number of towns	Number of roads
1	0
2	1
3	3
4	6
5	10
6	15
7	21
8	28
9	36
10	45
11	55
12	66
13	78
14	91
15	105

The number of roads for 5 towns helps you work out the number of roads for 6 towns.

Number of towns	Number of roads
5	10
6	5 + 10

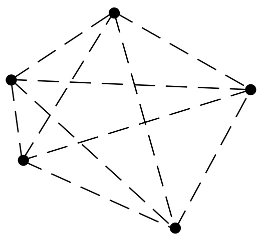
The number of roads for 13, 14, and 15 towns is also worked out using this pattern.

Number of towns	Number of roads
12	66
13	12 + 66 = 78

Number of towns	Number of roads
13	78
14	13 + 78 = 91

Number of towns	Number of roads
14	91
15	14 + 91 = 105

A rule for the number of roads for any number of towns can be worked out in the following way:



There are 5 towns, and there are 4 roads from each town, which is 5 x 4 = 20 roads. But each of these roads is also the road from another town, so there are just $^{1}/_{2}$ of 5 x 4 = 10 roads for 5 towns.

This can be generalised for any number of towns: the number of roads needed for any number of towns is one-half of the number of towns times 1 fewer than the number of towns.

We can express this symbolically: when there are n towns, there are $^{1}/_{2} \times n \times (n - 1)$ roads, or more simply, $^{1}/_{2}n(n - 1)$ or $\frac{n(n-1)}{2}$ roads. Note that n is often used when the quantity we are considering is a *whole* number. We could equally well have used x.

Page 17 Parking Fees

Achievement Objectives

- devise and use problem-solving strategies to explore situations mathematically (Mathematical Processes, problem solving, level 4)
- interpret information and results in context (Mathematical Processes, developing logic and reasoning, level 4)

ACTIVITY

This activity asks students to decide between various options for paying for car parking.

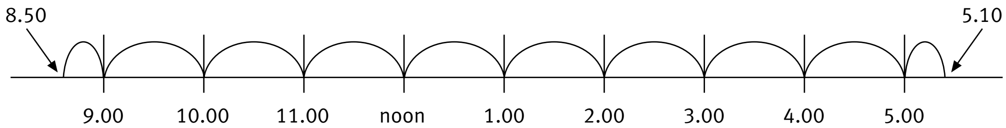
The students could use calendars to work out the number of work days that Peter might need a car park. They could shade those days when a car park will not be needed. These include Saturdays, Sundays, and national holidays. The calendar for January on the next page shows how this might work.

January

Monday		6	13	20	27
Tuesday		7	14	21	28
Wednesday	1	8	15	22	29
Thursday	2	9	16	23	30
Friday	3	10	17	24	31
Saturday	4	11	18	25	
Sunday	5	12	19	26	

This particular January has 21 days that Peter would need a car park if he were to work every day that is possible. It is likely that Peter will need a car park for fewer than 20 days in months where national holidays such as Waitangi Day or Easter occur.

Peter works from 9.00 a.m. to 5.00 p.m. but needs a car park from 8.50 a.m. to 5.10 p.m., allowing 10 minutes for walking to and from his office.



So he needs a car park for 8 hours, 20 minutes. But car parks are charged for each half-hour or part thereof, so Peter is charged for 8 hours, 30 minutes each day. This is the same as 17 half-hour periods. The first half-hour is charged at $2, and the remaining half-hours are charged at $1 each. So 17 half-hour periods will cost Peter 2 + 16 x 1 = $18. On Wednesdays, Peter needs a car park for an extra hour or 2 half-hours. This will cost an additional $2, making the charge for Wednesdays $20.

In question **2**, the students use the daily car-parking charges above to work out which is the better option. They see that a monthly charge of $320 will be cheaper as long as Peter works at least 18 days. (The actual savings will depend on the particular combination of Wednesdays and other days that Peter works in the month.)

Questions **3** and **4** are dealt with fully in the Answers. (Note that running costs on Peter's car while his sister is using it are not taken into account in question **4**.)

The Power of 2

Achievement Objectives

- devise and use problem-solving strategies to explore situations mathematically (Mathematical Processes, problem solving, level 4)
- find a rule to describe any member of a number sequence and express it in words (Algebra, level 4)
- use a rule to make predictions (Algebra, level 4)
- make conjectures in a mathematical context (Mathematical Processes, developing logic and reasoning, level 4)

ACTIVITY ONE

In this activity, the students should initially use geoboards or square dot paper to help them work out the area of the striped squares.

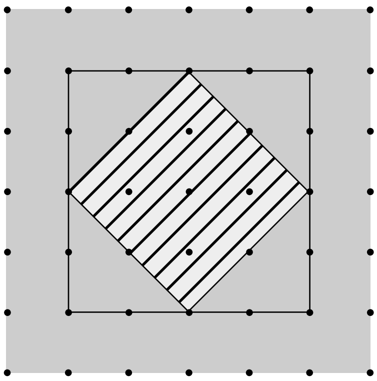


Figure 1

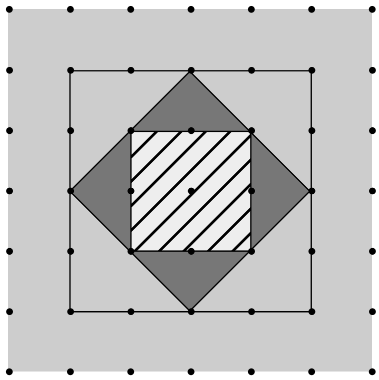
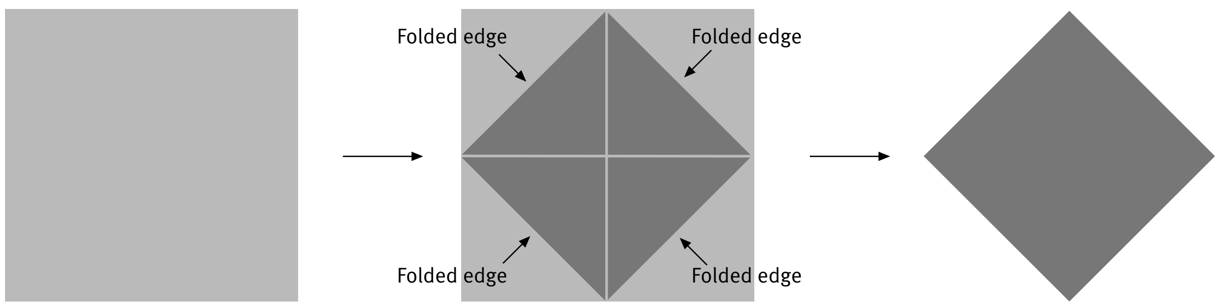


Figure 2

The area of the striped square in figure 1 (which is the shape in **iv**) is the area of the surrounding 4 by 4 square, that is, 16 square units, minus the area of the four corner triangles. The area of each triangle is 2 square units (one-half of a 2 by 2 square). So the striped area is equal to 16 square units minus 4 x 2 square units. We usually write 8 square units as 8 units^2, so in this case, the area is 16 units^2 – 8 units^2 = 8 units^2.

The area of the striped square in figure 2 (which is the shape in **iii**) is the area of the striped square from figure 1 minus the area of the four corner triangles. The area of each of these triangles is 1 unit^2 (two halves of a 1 by 1 square). So the striped area in figure 2 is equal to 8 units^2 – 4 x 1 = 4 units^2.

The students need to repeat this process for the other striped squares and write a rule connecting successive striped square areas. They will see that the striped area in figure 1 is double the striped area in figure 2. This relationship can also be seen clearly by folding squares of paper, as illustrated in the diagram below.



Square A Square B

Square A is twice the area of square B. Square A is the square that encloses square B.

So a simple rule is: the area of a square is twice the area of the square it encloses or, as given in the Answers, the area of a square is double the area of the enclosed square.

ACTIVITY TWO

In this activity, the students repeatedly fold and then cut pieces of paper. Each cut doubles the number of pieces of paper. These results can be shown in a table:

Number of cuts	Number of pieces of paper	Pattern
0	1	1
1	2	2
2	4	2 x 2 = 2^2
3	8	2 x 2 x 2 = 2^3
4	16	2 x 2 x 2 x 2 = 2^4
5	32	2 x 2 x 2 x 2 x 2 = 2^5
6	64	2 x 2 x 2 x 2 x 2 x 2 = 2^6
7	128	2 x 2 x 2 x 2 x 2 x 2 x 2 = 2^7

So, for 10 cuts, there will be 2^{10} pieces of paper. This is 2 x 2 x 2 x 2 x 2 x 2 x 2 x 2 x 2 x 2 = 1 024.
We express this as "2 to the power of 10 equals 1 024".

Pressing the following sequence of keys on most calculators will give the value of 2^{10} as 1 024.

$$\boxed{2}\ \boxed{\text{x}}\ \overbrace{\boxed{=}\ \boxed{=}\ \boxed{=}\ \boxed{=}\ \boxed{=}\ \boxed{=}\ \boxed{=}\ \boxed{=}\ \boxed{=}}^{9}$$

Where this sequence of keys doesn't work, experiment until you find a sequence that does. For example, the following sequence works with some calculators:

$$\boxed{2}\ \boxed{\text{x}}\ \boxed{\text{x}}\ \underbrace{\boxed{=}\ \boxed{=}\ \boxed{=}\ \boxed{=}\ \boxed{=}\ \boxed{=}\ \boxed{=}\ \boxed{=}\ \boxed{=}}_{9}$$

Scientific calculators can also be used to find the value of powers of numbers. We can express these algebraically as y^x, which we say as "any number, y, raised to the power of any number, x". So the value of 5^7 can be found by pressing the following keys:

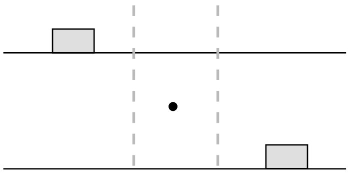
$$\boxed{5}\ \boxed{y^x}\ \boxed{7}\ \boxed{=}$$

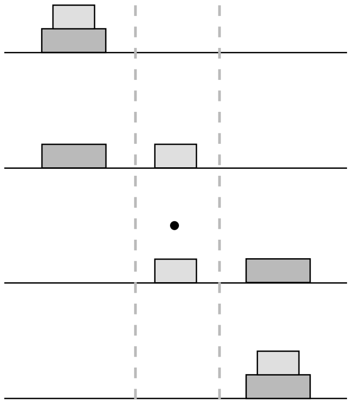
Note that some scientific calculators use the key x^y instead of y^x.

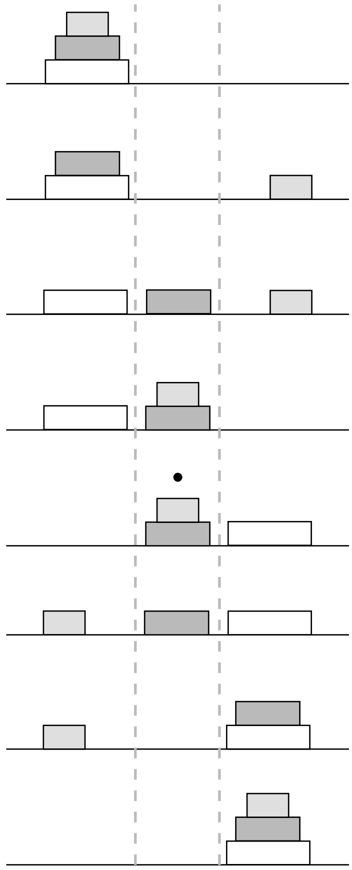
ACTIVITY THREE

This activity explores a simplified version of a mathematical puzzle that was invented in 1883 by the French mathematician Edouard Lucas (1842–1891). The puzzle is known as the Tower of Hanoi and sometimes as the Tower of Brahma. It was inspired by a Hindu legend that tells of the mental discipline demanded of young priests. The legend says that at the beginning of time, the priests in the temple were given 64 gold discs, each one a little smaller than the one beneath it. The priests were to transfer the 64 gold discs from one of three poles to another, via the second pole where necessary, in such a way that a disc could never be placed on top of a smaller disc. The legend goes on to say that when this task was finished, the temple would crumble into dust and the world would end.

The students are initially asked to see if they can work out the minimum number of moves to transfer only 3 lids (discs) rather than the 64 in the legend. To understand how this puzzle works, the students will find it helpful to try the puzzle with just 1 lid and then 2 lids. The following diagrams show the moves for 1, 2, and 3 lids.


1 lid, 1 move


2 lids, 3 moves


3 lids, 7 moves

If you look closely at the diagrams, you may notice a symmetrical pattern. Each diagram has a centre of rotational symmetry, marked with a • .

The pattern for the number of moves is shown in this table:

Number of lids	Number of moves	Pattern for number of moves
1	1	1
2	3	$2 \times 2 - 1 = 2^2 - 1$
3	7	$2 \times 2 \times 2 - 1 = 2^3 - 1$
4	15	$2 \times 2 \times 2 \times 2 - 1 = 2^4 - 1$
5	31	$2 \times 2 \times 2 \times 2 \times 2 - 1 = 2^5 - 1$
6	63	$2 \times 2 \times 2 \times 2 \times 2 \times 2 - 1 = 2^6 - 1$
7	127	$2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 - 1 = 2^7 - 1$

The pattern in the table shows that there are $2^{(\text{number of lids})} - 1$ moves for any number of lids. When there are x lids, the number of moves can be expressed algebraically as $2^x - 1$.

If we return to the Hindu legend associated with this puzzle, we see that to transfer 64 gold discs, $2^{64} - 1$ moves will be required. Altogether, this is 18 446 744 073 709 551 615 moves. If the priests worked continuously for 24 hours a day, 7 days a week, making one move every second, the complete transfer would take slightly more than 580 billion years. This is more than current estimates for the age of the universe, so the legend may well be correct in asserting that the world will end when the priests finish their task!

Page 20 Skateboardathon

Achievement Objectives

- find and justify a word formula which represents a given practical situation (Algebra, level 4)
- find a rule to describe any member of a number sequence and express it in words (Algebra, level 4)
- use a rule to make predictions (Algebra, level 4)
- use words and symbols to describe and generalise patterns (Mathematical Processes, developing logic and reasoning, level 4)

ACTIVITY

In this activity, students use tables to organise data taken from a real-life situation. They look for patterns in the data to help them devise a rule for making sensible predictions related to raising money from the skateboardathon.

In question **1**, the numbers in the running total column in the table form a simple pattern involving squares of the corresponding numbers in the hours column. The rule is: the amount raised by each boy is the square of the number of hours skated. Using algebra, if n is the number of hours on the skateboard, the value of the amount raised is n^2 dollars. This corresponds to a simple but elegant mathematical rule: the sum of the first n odd numbers equals n squared, for example, $1 + 3 + 5 + 7 = 16$, which is 4^2.

In question **2**, there are two patterns that the students might recognise for the numbers in the running total column. It may be easier for the students to find these patterns if they compare the values in this table with those in the table for question **1**.

Hours	Amount in $	Running total in $	First pattern	Second pattern
1	2	2	1 x 1 + 1	1 x 2
2	4	6	2 x 2 + 2	2 x 3
3	6	12	3 x 3 + 3	3 x 4
4	8	20	4 x 4 + 4	4 x 5
5	10	30	5 x 5 + 5	5 x 6
6	12	42	6 x 6 + 6	6 x 7
7	14	56	7 x 7 + 7	7 x 8
8	16	72	8 x 8 + 8	8 x 9
9	18	90	9 x 9 + 9	9 x 10
10	20	110	10 x 10 + 10	10 x 11

A rule for the first pattern is: the amount raised by each boy is the square of the number of hours skated plus the number of hours skated. In algebra, if the skateboarding lasted for n hours, the value of the amount raised is $n^2 + n$ dollars. Note that this rule is n dollars more than in the rule for question **1**.

A rule for the second pattern is: the amount raised by each boy is the product of the number of hours of skateboarding and 1 more than the number of hours of skateboarding. So if the skateboarding lasted for n hours, the value of the amount raised is $n \times (n + 1)$ dollars. This algebraic rule is usually expressed as $n(n + 1)$. Using the rule for the first pattern gives 24 x 24 + 24 = $600. The second rule also gives $600 from the calculation 24 x 25. Algebraically, $n(n + 1)$ can be expanded to give $n^2 + n$, so the two rules are equivalent.

Digit Chains

Achievement Objectives

- devise and follow a set of instructions to carry out a mathematical activity (Mathematical Processes, communicating mathematical ideas, level 4)
- use a rule to make predictions (Algebra, level 4)
- find and justify a word formula which represents a given practical situation (Algebra, level 4)
- use words and symbols to describe and generalise patterns (Mathematical Processes, developing logic and reasoning, level 4)
- find a rule to describe any member of a number sequence and express it in words (Algebra, level 4)

ACTIVITY

This activity focuses on digit rules to form digit chains. In each chain, a rule is used to change one number into another.

In question **1**, a table like the one below may help the students to identify the digit rule. They should focus on the number of tens and ones to see if they can find a link between these values and the new number in each case. (If they follow the hint and look at the rules for questions **2** and **3**, they will see that an operation such as addition or multiplication is carried out on one or more of the digits in each number.)

Number	Tens and ones	New number	Pattern
93	9 tens and 3 ones	15	9 + 3 + ❐
15	1 ten and 5 ones	11	1 + 5 + ❐
11	1 ten and 1 one	3	1 + 1 + ❐
3	0 tens and 3 ones	6	0 + 3 + ❐
6	0 tens and 6 ones	12	0 + 6 + ❐
12	1 ten and 2 ones	5	1 + 2 + ❐

The values that go in the empty boxes in the table are the values for the number of ones. So the new number 15 is found from 9 + 3 + 3, the new number 11 is found from 1 + 5 + 5, and so on. The digit rule is: double the ones digit and add the tens digit.

The starting number, 93, becomes 15 where it enters the loop. A diagram of the loop is shown in the Answers.

Notice that the loop includes all the numbers from 1 to 18. The number 19 is not in the loop. It stays as 19 when the digit rule is applied: 19 $\longrightarrow$ 1 + 2 x 9 = 19
The next number, 20, enters the loop at 2, the number 21 enters at 4, the number 22 enters at 6, and so on. Multiples of 19, however, all eventually produce 19 when the digit rule is applied. For example, 95 = 19 x 5, so it is a multiple of 19. Applying the digit rule gives the following: 95 $\longrightarrow$ 9 + 2 x 5 = 19

The number 399 is also a multiple of 19 because 399 = 19 x 21. The digit rule is applied to tens and ones, so we think of 399 as 39 tens and 9 ones rather than as 3 hundreds, 9 tens, and 9 ones. The digit rule then produces the following: 399 $\longrightarrow$ 39 + 2 x 9 = 57 $\longrightarrow$ 5 + 2 x 7 = 19

So multiples of 19 do not enter the loop.

In question **2b**, the students will find that 2-digit numbers reduce to a single-digit number from 1 to 9.
For example, three successive applications of the digit rule reduce 67 to 4. Note that there is no change when the digit rule is applied to the single digit 4.
67 $\longrightarrow$ 6 x 3 + 7 = 25 $\longrightarrow$ 2 x 3 + 5 = 11 $\longrightarrow$ 1 x 3 + 1 = 4 $\longrightarrow$ 0 x 3 + 4 = 4

The first 3-digit number is 100, which is 10 tens and 0 ones. So 100 becomes 10 x 3 + 0 = 30. This reduces to 3 x 3 + 0 = 9. The 3-digit number 263 = 26 tens and 3 ones. So 263 becomes 26 x 3 + 3 = 81. This reduces to 8 x 3 + 1 = 25, which reduces to 2 x 3 + 5 = 11, which in turn reduces to 1 x 3 + 1 = 4 (as per the example for 67 above). 3-digit numbers all reduce to 2-digit numbers when the rule is applied, so they also reduce to single-digit numbers from 1 to 9.

4-digit numbers also change in the same way. For example, 2 317 = 231 tens and 7 ones. So 2 317 becomes 231 x 3 + 7 = 700. This number reduces to 70 x 3 + 0 = 210, which then reduces to 21 x 3 + 0 = 63. Further use of the rule gives 6 x 3 + 3 = 21, which reduces to 2 x 3 + 1 = 7. So, just as with 3-digit numbers, all 4-, 5-, and 6-digit numbers and so on reduce to a single-digit number from 1 to 9.

In question **3**, the loop is never ending, so any number that enters the loop can be changed into any number in the loop. For example, 43 changes to 35, which changes to 40, then 20, then 10, then 5, then 25, and then 35, where it first entered the loop. Zara concluded that since all four of her chosen starting numbers entered the loop, then all numbers enter the loop. In fact, she is not correct. Multiples of 3 (that is, 3, 6, 9, 12, …) don't enter the loop. Those that are also multiples of 9 (that is, 9, 18, 27, 36, …) become 45, which itself becomes 45 because this is a loop with only one value, and the others enter a loop that includes only 30 and 15. Notice how 30 changes to 15, which then changes back to 30:

30 ⟶ 3 x 5 + 0 x 5 = 15 ⟶ 1 x 5 + 5 x 5 = 30

Pages 22–23 Number Puzzles

Achievement Objectives

- use words and symbols to describe and generalise patterns (Mathematical Processes, developing logic and reasoning, level 4)
- devise and follow a set of instructions to carry out a mathematical activity (Mathematical Processes, communicating mathematical ideas, level 4)
- pose questions for mathematical exploration (Mathematical Processes, problem solving, level 4)
- devise and use problem-solving strategies to explore situations mathematically (Mathematical Processes, problem solving, level 4)
- solve simple linear equations such as 2❐ + 4 = 16 (Algebra, level 4)

ACTIVITY

In this activity, students show how some simple number puzzles work by writing sets of instructions using algebraic symbols.

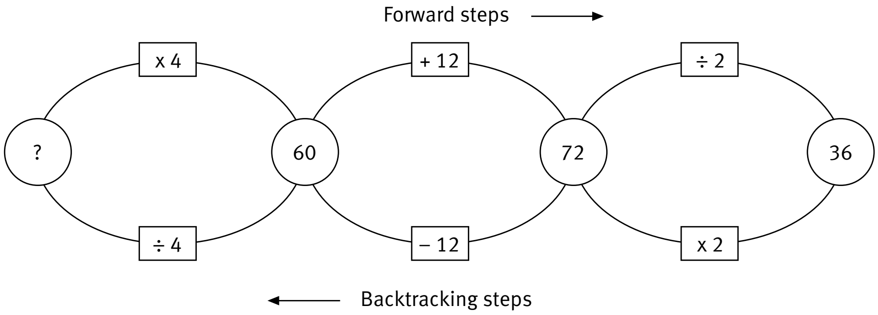
For example, in question **1a**, instead of thinking of a particular number, say 43, we can use *n* to stand for *any* number.

Think of a number.	⟶	Start with *n*	n
Add 22.		Add 22	$n + 22$
Subtract 6.		Subtract 6	$n + 16$
Multiply by 2.		Multiply by 2	$2 \times n + 32$
Subtract 8.		Subtract 8	$2 \times n + 24$
Halve the answer.		Halve the answer	$n + 12$

Note that when the expression $n + 16$ is multiplied by 2, we get $2 \times (n + 16) = 2 \times n + 2 \times 16$ or $2n + 32$. The value of each number or letter is doubled. Similarly, when $2n + 24$ is halved, we get $n + 12$, which is 12 more than the value for n, the starting number.

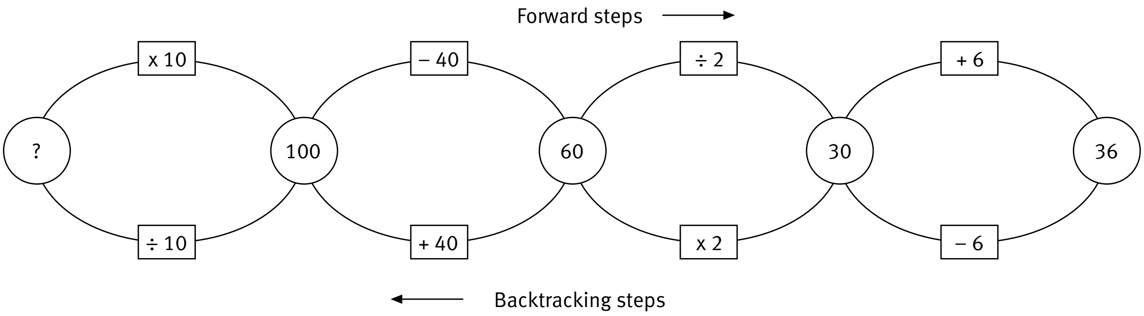
Students who have difficulty with these puzzles may find it helpful to work through them with numbers in order to gain confidence with the process.

In question **2**, the students need to work backwards, undoing each step as they go. The backtracking steps for question **2a** are shown in this diagram:



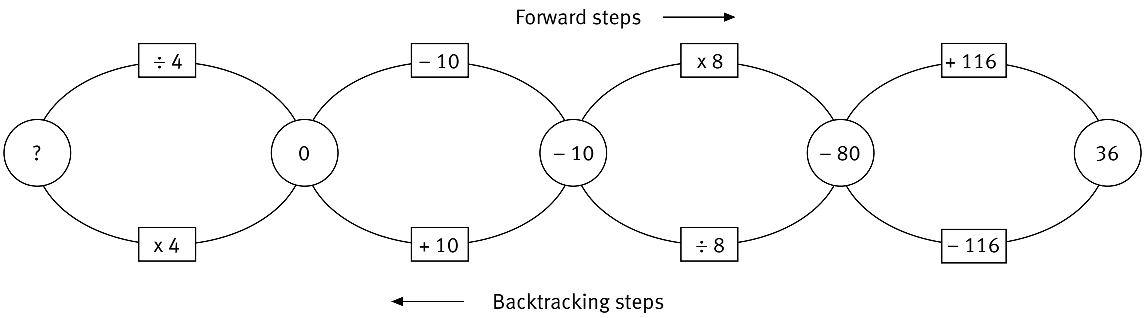
The numbers in the circles are found by working backwards from the answer, 36. The starting number, 15, is found from the final backtracking step, $60 \div 4$.

The backtracking diagram for question **2b** is:



The final backtracking step, $100 \div 10$, gives the starting number, 10.

The backtracking diagram for question **2c** is:



The final backtracking step, $0 \div 4$, gives the starting number, 0.

The backtracking diagrams are procedures that provide a careful, systematic approach to “undoing” the puzzles. The students will benefit most, however, from exploring the puzzles themselves and trying out their own strategies. It may be that their strategies reflect the backtracking ideas shown above.

More able students might be interested to see an algebraic approach to these problems. Such an approach could use the sets of instructions used in questions **1** and **2**. For example, in question **2a,** we have the instructions:

start with n — n
multiply by 4 — $4 \times n$
add 12 — $4 \times n + 12$
divide by 2 — $2 \times n + 6$.

So every starting number ends up being multiplied by 2 and having 6 added. To find the number n so that $2 \times n + 6 = 36$, we can use the following steps:

If $2 \times n + 6 = 36$
then $2 \times n = 30$ (subtracting 6 from both sides)
then $n = 15$ (dividing both sides by 2).

In question **2b**, the students will need to find n when $5n - 14 = 36$ ($n = 10$). In question **2c**, the students will need to find n when $2n + 36 = 36$ ($n = 0$).

Page 24 Changing Tyres

Achievement Objectives

- use words and symbols to describe and generalise patterns (Mathematical Processes, developing logic and reasoning, level 4)
- find and justify a word formula which represents a given practical situation (Algebra, level 4)
- sketch and interpret graphs on whole number grids which represent simple everyday situations (Algebra, level 4)
- interpret information and results in context (Mathematical Processes, developing logic and reasoning, level 4)
- use a rule to make predictions (Algebra, level 4)

ACTIVITY

In this activity, students reason through practical situations to enable them to write algebraic formulae representing the situations.

The table below may help the students to see the relationship between the number of trucks and the number of tyres that should be budgeted for annually.

Number of trucks	Number of tyres per 6 months	Number of tyres per year
1	14	28 = 28 x 1
2	28	56 = 28 x 2
3	42	84 = 28 x 3
4	56	112 = 28 x 4
5	70	140 = 28 x 5
6	84	168 = 28 x 6
7	98	196 = 28 x 7
8	112	224 = 28 x 8
t		$n = 28 \times t$

The pattern shows that if the number of tyres to be budgeted for annually is n and the number of trucks owned is t, the formula $n = 28 \times t$ (or $n = 28t$) can be used to work out the annual truck tyre budget.

The following table shows a similar relationship for budgeting for the annual purchase of car tyres. Note that car tyres, on average, last 50 000 kilometres and, on average, cars travel 20 000 kilometres annually, so it takes, on average, 50 000 km ÷ 20 000 km per year = 2.5 years before car tyres should be changed.

Number of cars	Number of tyres per 2.5 years	Number of tyres per year	Pattern for number of tyres per year
1	5	5 ÷ 2.5 = 2	2 = 2 x 1
2	10	10 ÷ 2.5 = 4	4 = 2 x 2
3	15	15 ÷ 2.5 = 6	6 = 2 x 3
4	20	20 ÷ 2.5 = 8	8 = 2 x 4
5	25	25 ÷ 2.5 = 10	10 = 2 x 5
6	30	30 ÷ 2.5 = 12	12 = 2 x 6
7	35	35 ÷ 2.5 = 14	14 = 2 x 7
8	40	40 ÷ 2.5 = 16	16 = 2 x 8
c			$n = 2 \times c$

The pattern in the final column shows that, if the number of tyres to be budgeted for annually is n and the number of cars owned is c, then the formula $n = 2 \times c$ (or $n = 2c$) can be used to work out the annual car tyre budget.

Some students may have difficulty calculating that, on average, Mik's dad will need 2 new tyres for each car per year. The following argument may help: Consider just one car and assume that Mik's dad changes all the tyres at the same time. Each tyre lasts for 2.5 years, so every 2.5 years, Mik's dad will need to change all 5 tyres. Therefore, every 5 years, Mik's dad will need 10 new tyres. This is the same as buying 2 new tyres each year.

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